JOURNAL OF FUNCTIONAL ANALYSIS **134**, 350–362 (1995)

Every Infinite-Dimensional Hilbert Space is Real-Analytically Isomorphic with Its Unit Sphere

TADEUSZ DOBROWOLSKI*

Department of Mathematics, The University of Oklahoma, 601 Elm Avenue, Room 423, Norman, Oklahoma 73109-0315

Communicated by A. Connes

Received January 24, 1994

DEDICATED TO CZESŁAW BESSAGA ON HIS SIXTIETH BIRTHDAY
FOR TEACHING ME TO LOVE MATHEMATICS

The title statement is proved. Similar results for arbitrary Banach spaces are obtained in both the real-analytic and the C^p settings.

1. INTRODUCTION

Let $E = (E, \|\cdot\|)$ be a Banach space and let ω be a continuous norm on E (i.e., $\omega(x) \leqslant C\|x\|$ for some constant $C > 0$ and all $x \in E$). Assume that ω is real-analytic (resp., of class C^p, $p = 1, 2, ..., \infty$), that is, ω is real-analytic (resp., of class C^p) on $E \backslash \{0\}$. Let $S_\omega = \{x \in E \mid \omega(x) = 1\}$ be the ω-unit sphere. Then S_ω carries the natural real-analytic (resp., C^p) manifold structure determined as follows. For every $z \in S_\omega$, let P_z be the hyperplane tangent to S_ω at $-z$. Write π_z for the stereographic projection of $S_\omega \backslash \{z\}$ onto P_z. The desired manifold structure on S_ω is defined by the family $\{\pi_z \mid z \in S_\omega\}$. It can be checked that S_ω is a real-analytic (resp., C^p) submanifold, modelled on a codimension one, closed linear subspace E_0 of E; cf. [Lang, II, §2, Example]. (The space E_0 can be identified with $\ker x^*$, where $0 \neq x^* \in E^*$.) If E is infinite-dimensional, then the homotopy types of S_ω and of E_0 coincide and, applying results of infinite-dimensional topology, S_ω is homeomorphic to E_0. The question arises whether S_ω and E_0 are real-analytically (resp., C^p) isomorphic.

This paper answers that question affirmatively in the case of separable E. Since, obviously, a Hilbertian norm is real-analytic then, in particular, we get

* Current address: Department of Mathematics, Pittsburg State University, Pittsburg, Kansas 66762. E-mail address: tdobrowo@mail.pittstate.edu.

0022-1236/95 $12.00

THEOREM. *Let H be an infinite-dimensional Hilbert space and let S be the unit sphere of H. There exists a real-analytic isomorphism f from S onto H.*

This provides an affirmative answer to Bessaga's Problem 6 in [Bes], and extends his theorem therein stating that S and H are C^∞ isomorphic. Let us recall that Bessaga's theorem was an important ingredient in the theory of infinite-dimensional C^∞ Hilbert manifolds. The main results of this theory state that: (1) the homotopy and the C^∞ isomorphism classifications of such manifolds coincide (see [BK]), and (2) every such manifold admits a C^∞ open embedding into H [EE]. It would be interesting to extend the results (1) and (2) to the real-analytic category. Our result can be viewed as a small move towards this (obviously, the basic difference between the C^∞ and the real-analytic categories is the lack of real-analytic partitions of unity in the real-analytic setting).

Amazingly enough not only do we show that there exists a real-analytic isomorphism $f: S \to H$ but, in fact, we give an explicit formula for f. The key ingredient of our approach is Bessaga's incomplete norm technique which we adapt here to delete one-point sets from the sphere S in a real-analytic way. Without going into details, let us say that we replace the natural affine action of H on H by the (natural) action of the group of isometries of H on S (cf. [Dob2]).

As a by-product of our approach we construct a C^∞-isomorphism of $H\backslash\{z\}$ onto H, $\|z\| = 1$, which preserves all concentric spheres (and, in fact, can be identity off an arbitrary neighborhood of S). Moreover, $\{z\}$ can be replaced by an arbitrary compact set $K \subset S$. This shows that the unit closed ball B, a C^∞-manifold having S as its boundary, is C^∞ isomorphic to $B\backslash K$ whose boundary is just $S\backslash K$.

Some results concerning arbitary Banach (including nonseparable) spaces are also discussed along these lines.

2. THE PROOF OF THEOREM

We start with the following elementary observation.

LEMMA 1. *Let E be a Banach space, and let ω_1 and ω_2 be real-analytic (resp., of class C^p, $p = 1, 2, \ldots, \infty$) norms on E. The formula*

$$h_1(x) = \frac{x}{\omega_2(x)}, \qquad x \in E$$

establishes a real-analytic (resp., C^p) isomorphism of S_{ω_1} onto S_{ω_2}.

Proof. Since S_{ω_1} and S_{ω_2} are submanifolds of E, it is enough to notice that h_1 and $h_1^{-1}(y) = y/\omega_1(y)$, $y \in S_{\omega_2}$, are real-analytic (resp., of class C^p) on some neighborhoods of S_{ω_1} and S_{ω_2}, respectively. ∎

Let $T: E \to H$ be a continuous injective operator of a Banach space $E = (E, \|\cdot\|)$ into a Hilbert space $(H, \langle \cdot, \cdot \rangle)$. For every $x, y \in E$, we write

$$\langle x, y \rangle_\omega = \langle Tx, Ty \rangle.$$

Then $(E, \langle \cdot, \cdot \rangle_\omega)$ is a pre-Hilbert space whose completion will be denoted by $\hat{E}_\omega$; clearly, both the inner product $(x, y) \to \langle x, y \rangle_\omega$ and the norm $x \to \omega(x) = \sqrt{\langle x, x \rangle_\omega}$ are real-analytic functions on $\hat{E}_\omega$. We let

$$\hat{S}_\omega = \{x \in \hat{E}_\omega \mid \omega(x) = 1\},$$

and

$$S_\omega = \{x \in E \mid x \in \hat{S}_\omega\}.$$

We will consider E as a dense subspace of $\hat{E}_\omega$; it follows that S_ω is dense in $\hat{S}_\omega$.

PROPOSITION 1. *Suppose that there exists a path* p: $[0, t_0) \to \hat{S}_\omega$, $t_0 > 0$, *satisfying the following conditions*:

(a) $p(t) \in S_\omega$ *for* $0 < t < t_0$, *and* $p(0) \in \hat{S}_\omega \backslash S_\omega$;

(b) $p \mid (0, t_0)$: $(0, t_0) \to S_\omega$ *is real-analytic as a map into* $(E, \|\cdot\|)$;

(c) *there exists a constant* $M > 0$ *such that* $\omega(p(t) - p(s)) \leqslant M\,|t - s|$, $0 \leqslant t, s \leqslant t_0$.

Fix an arbitrary $z \in S_\omega$ *so that* $\langle z, p(0) \rangle_\omega \neq 0$. *Let* $0 < L < \min\{t_0/2, 1/8M\}$, *and write* $d(x) = L\omega(x - z)$. *Then the formula*

$$h(x) = x - 2\langle x, p(d(x)) \rangle_\omega \cdot p(d(x)), \qquad x \in S_\omega \backslash \{z\}$$

establishes a real-analytic isomorphism of $S_\omega \backslash \{z\}$ *onto* S_ω.

LEMMA 2. *For every* $x \in \hat{S}_\omega$, *let* $A = A(x)$ *be given by*

$$A(y) = y - 2\langle y, x \rangle_\omega \cdot x.$$

Then, we have

(a) A: $\hat{E}_\omega \xrightarrow{\text{onto}} \hat{E}_\omega$ *is an isometry such that* A^2 *is the identity operator*;

(b) *if* $x_0 \in S_\omega$, *then* $A \mid E$ *is a continuous invertible operator of* E *onto* E; *in particular*, $A(S_\omega) = S_\omega$ *and* $A(\hat{S}_\omega \backslash S_\omega) \subset \hat{S}_\omega \backslash S_\omega$;

(c) $\omega(A(x)(y) - A(x')(y)) \leqslant 2\omega(x - x')$ *for every* $x, x', y \in \hat{S}_\omega$.

Proof. Since $\omega^2(x)=1$, we have

$$\begin{aligned}\omega^2(A(y)) &= \langle y-2\langle y,x\rangle_\omega\cdot x,\ y-2\langle y,x\rangle_\omega\cdot x\rangle_\omega\\ &= \langle y,y\rangle_\omega-2\langle y,x\rangle_\omega^2-2\langle y,x\rangle_\omega^2+4\langle y,x\rangle_\omega^2\langle x,x\rangle_\omega\\ &= \langle y,y\rangle_\omega=\omega^2(y).\end{aligned}$$

If $z=A(y)=y-2\langle y,x\rangle_\omega\cdot x$, $z\in\hat{S}_\omega$, then $\langle z,x\rangle_\omega=\langle y,x\rangle_\omega-2\langle y,x\rangle_\omega = -\langle y,x\rangle_\omega$; consequently, $z=y+2\langle z,x\rangle_\omega\cdot x$. Hence, we have that $y=z-2\langle z,x\rangle_\omega\cdot x$ and so $A^{-1}(z)=A(z)$. The item (a) is shown.

The item (b) is obvious. To show (c), let $x,x',y\in\hat{S}_\omega$ and estimate

$$\begin{aligned}\omega(A(x)(y)-A(x')(y)) &= \omega(\langle y,x\rangle_\omega\cdot x-\langle y,x'\rangle_\omega\cdot x')\\ &\leqslant \omega(\langle y,x\rangle_\omega\cdot x-\langle y,x'\rangle_\omega\cdot x)+\omega(\langle y,x'\rangle_\omega\cdot x\\ &\quad -\langle y,x'\rangle_\omega\cdot x')\\ &\leqslant |\langle y,x-x'\rangle_\omega|\,\omega(x)+|\langle y,x'\rangle_\omega|\,\omega(x-x')\\ &\leqslant \omega(y)\,\omega(x-x')+\omega(y)\,\omega(x')\,\omega(x-x')\\ &= 2\omega(x-x'). \quad \blacksquare\end{aligned}$$

Proof of Proposition 1. Observe that there exists $\varepsilon>0$ so that $h(x)$ makes sense for every x in $U_\varepsilon=\{x\in\hat{E}_\omega \mid |\omega(x)-1|<\varepsilon\}$. Since $h(x)=A(p(d(x)))(x)$, by Lemma 2(a), $h: U_\varepsilon\to U_\varepsilon$, moreover, $\omega(h(x))=\omega(x)$.

We will show that $h: \hat{S}_\omega\to\hat{S}_\omega$ is a bijection (in general, h is bijection on U_ε for some $\varepsilon>0$). To this end, fix $y\in\hat{S}_\omega$ and designate $\Phi: \hat{S}_\omega\to\hat{S}_\omega$ by

$$\Phi(x)=A(\varphi(x))(y), \qquad x\in\hat{S}_\omega,$$

where $\varphi(x)=p(d(x))$. By our assumptions, we can estimate as follows

$$\begin{aligned}\omega(\varphi(x)-\varphi(x')) &\leqslant M\,|d(x)-d(x')|\leqslant ML\omega(x-x')\\ &\leqslant \tfrac{1}{8}\omega(x-x').\end{aligned}$$

This together with Lemma 2(c) yields

$$\begin{aligned}\omega(\Phi(x)-\Phi(x')) &= \omega(A(\varphi(x))(y)-A(\varphi(x'))(y))\\ &\leqslant 2\omega(\varphi(x)-\varphi(x'))\leqslant 2\cdot\tfrac{1}{8}\omega(x-x')=\tfrac{1}{4}\omega(x-x')\end{aligned}$$

for every $x,x'\in\hat{S}_\omega$. Applying the Banach Contraction Principle to the complete metric space $\hat{S}_\omega$ and to the map $\Phi: \hat{S}_\omega\to\hat{S}_\omega$, we infer that there exists a unique $x \in \hat{S}_\omega$ such that $x = \Phi(x) = A(\varphi(x))(y)$. Using Lemma 2(a), we equivalently have that

$$h(x) = A(\varphi(x))(y) = A^2(\varphi(x))(y) = y.$$

The same argument shows that $h\colon U_\varepsilon \to U_\varepsilon$ is a bijection for some $\varepsilon > 0$.

We claim that $h(S_\omega \backslash \{z\}) = S_\omega$. Suppose $x \in S_\omega \backslash \{z\}$. It follows that $d(x) > 0$. By the assumption (a), $\varphi(x) = p(d(x)) \in S_\omega$; applying Lemma 2(b), $h(x) = A(\varphi(x))(x) \in S_\omega$. Now suppose that $h(x) = y$ for some $y \in S_\omega$. Then x cannot be in $\hat{S}_\omega \backslash S_\omega$; otherwise we would have $d(x) > 0$, and since $\varphi(x) = p(d(x)) \in S_\omega$, by Lemma 2(b), $h(x) \in \hat{S}_\omega \backslash S_\omega$, a contradiction. Moreover, x cannot be z; otherwise $\varphi(z) = p(0)$ and hence $y = h(z) = A(p(0))(z) = z - 2\langle z, p(0)\rangle_\omega \cdot p(0) \in \hat{S}_\omega \backslash S_\omega$ because $\langle z, p(0)\rangle_\omega \neq 0$ and $p(0) \in \hat{S}_\omega \backslash S_\omega$.

We have $h(x) = x - 2\langle x, p(d(x))\rangle_\omega \cdot p(d(x))$, $x \in S_\omega \backslash \{z\}$. Since d is real-analytic on $S_\omega \backslash \{z\}$ and $p \mid (0, t_0)$, as a map into E, is real-analytic as well, h is also real-analytic. To show the real-analyticity of h^{-1}, consider

$$H(x, y) = x - A(\varphi(x))(y)$$

defined on $U_\varepsilon \times U_\varepsilon$ into $\hat{E}_\omega$. Below we will show that for every pair $(x, y) \in S_\omega \times S_\omega$, $x \neq z$, the operator

$$T = \frac{\partial H(x, y)}{\partial x} : E \to E$$

is invertible. Then, by an application of the implicit function theorem (cf. [Die, 10.2.5] and [Wh]) we conclude that h^{-1}, which satisfies $H(h^{-1}(y), y) = 0$, is real-analytic when restricted to S_ω.

Since $H(x, y) = x + 2\langle y, \varphi(x)\rangle_\omega \cdot \varphi(x) - y$, it follows that $T(v) = v + 2D[\langle x, \varphi(x)\rangle_\omega \cdot \varphi(x)](v)$, where D stands for the derivative operator. We see that

$$\begin{aligned} D[\langle y, \varphi(x)\rangle_\omega](v) \cdot \varphi(x) + \langle y, \varphi(x)\rangle_\omega \cdot D[\varphi(x)](v) \\ = \langle y, D[\varphi(x)](v)\rangle_\omega \cdot \varphi(x) + \langle y, \varphi(x)\rangle_\omega \cdot D[\varphi(x)](v). \end{aligned}$$

From (c) we get $D[\varphi(x)](v) = v_0 D[d(x)](v)$, where $v_0 \in E$ with $\omega(v_0) \leqslant M$. Moreover, we have $\sup\{|D[d(x)](v)| \mid \omega(v) \leqslant 1\} \leqslant L$. It follows that $\sup\{\omega(D[\varphi(x)](v)) \mid \omega(v) \leqslant 1\} \leqslant LM \leqslant \frac{1}{8}$, and consequently $\sup\{2\omega(D[\langle y, \varphi(x)\rangle_\omega \cdot \varphi(x)](v)) \mid \omega(v) \leqslant 1\} \leqslant 2(\frac{1}{8} + \frac{1}{8}) = \frac{1}{2}$. We can now easily conclude that T is invertible. The proof is complete. ∎

Remark 1. The above argument shows that

(a) there exists $\varepsilon > 0$ so that h establishes a real-analytic isomorphism of $\{x \in E \mid |\omega(x) - 1| < \varepsilon\} \backslash \{z\}$ onto $\{x \in E \mid |\omega(x) - 1| < \varepsilon\}$ that preserves the concentric spheres;

(b) with such ε, $h: U_\varepsilon \backslash \{z\} \xrightarrow{\text{onto}} U_\varepsilon \backslash \{h(z)\}$ is a real-analytic isomorphism;

(c) $h: \hat{S}_\omega \to \hat{S}_\omega$ is a homeomorphism, and both h and h^{-1} are Lipschitz.

To justify (c), let $x = h^{-1}(y)$ and $x' = h^{-1}(y')$. We have

$$\begin{aligned} \omega(x - x') &= \omega(A(\varphi(x))(y) - A(\varphi(x))(y)) \\ &\leqslant \omega(A(\varphi(x))(y) - A(\varphi(x))(y')) + \omega(A(\varphi(x))(y') - A(\varphi(x'))(y')) \\ &\leqslant \omega(y - y') + \tfrac{1}{4}\omega(x - x'). \end{aligned}$$

We see that $\frac{3}{4}\omega(h^{-1}(y) - h^{-1}(y')) \leqslant \omega(y - y')$.

Remark 2. Suitable C^p versions of Proposition 1 and Remark 2 hold true.

Below we will give an explicit formula for $p(t)$ in the case of $E = l^2$. Here we will consider $l^2 = \{x = (x_n)_{n=0}^\infty \mid \sum_{n=0}^\infty x_n^2 < \infty\}$ with the norm $\|x\| = \sqrt{\sum_{n=0}^\infty x_n^2}$. Let $T: l^2 \to l^2$ be given by

$$T(x) = \left(\frac{x_n}{2^n}\right), \qquad x = (x_n)_{n=0}^\infty \in l^2.$$

The space $\hat{l}^2_\omega$ can be identified with $\{x = (x_n)_{n=0}^\infty \mid \sum_{n=0}^\infty (x_n/2^n)^2 < \infty\}$ equipped with the inner product $\langle x, y \rangle_\omega = \sum_{n=0}^\infty x_n y_n / 2^{2n}$.

LEMMA 3. *There exists a path $p: [0, 1) \to \hat{S}_\omega$ fulfilling the items* (a)–(c) *of Proposition* 1.

Proof. For $0 \leqslant t \leqslant 1$, we let $q(t) = (1, t, t^2, ...) \in \hat{l}^2_\omega$. We have that

$$\omega^2(q(t)) = \sum_{n=0}^\infty \left(\frac{t^n}{2^n}\right)^2 = \sum_{n=0}^\infty \left[\left(\frac{t}{2}\right)^2\right]^n = \frac{1}{1 - t^2/4} = \frac{4}{4 - t^2}$$

for $0 \leqslant t \leqslant 1$. We let $p_0(t) = (\sqrt{(4 - t^2)}/2)\, q(t)$, $0 \leqslant t \leqslant 1$, and set

$$p(t) = p_0(1 - t), \qquad 0 \leqslant t \leqslant 1.$$

Clearly, $p\colon [0, 1] \to \hat{S}_\omega$ and $p(0) = p_0(1) = (\sqrt{3}/2)\, q(1) \in \hat{S}_\omega \backslash S_\omega$. Moreover, $p \mid (0, 1) : (0, 1) \to S_\omega \subset l^2$ is real-analytic. It remains to establish the item (c) of Proposition 1. To this end, it is enough to show that the derivative of p_0, as a map into $\hat{l}^2_\omega$, has bounded norm.

We have

$$D[p_0(t)] = \left(\sqrt{1 - \frac{t^2}{4}}\right)' q(t) + \sqrt{1 - \frac{t^2}{4}}\, D[q(t)]$$
$$= \frac{t}{2\sqrt{4 - t^2}}\, q(t) + \sqrt{1 - \frac{t^2}{4}}\, D[q(t)];$$

consequently,

$$\omega(D[p_0(t)]) \leqslant \frac{t}{2\sqrt{4 - t^2}}\, \omega(q(t)) + \sqrt{1 - \frac{t^2}{4}}\, \omega(D[q(t)])$$
$$\leqslant \frac{t}{2\sqrt{4 - t^2}} \cdot \frac{2}{\sqrt{4 - t^2}} + \sqrt{1 - \frac{t^2}{4}}\, \omega(D[q(t)])$$
$$= \frac{t}{4 - t^2} + \sqrt{1 - \frac{t^2}{4}}\, \omega(D[q(t)]).$$

For $0 \leqslant t \leqslant 1$, we have $\omega^2(D[q(t)]) = \sum_{n=1}^{\infty} (nt^{n-1}/2^n)^2 \leqslant \sum_{n=1}^{\infty} nt^{n-1}/2^n = 2/(2 - t)^2$. This together with the above yields

$$\omega(D[p_0(t)]) \leqslant \frac{t}{4 - t^2} + \frac{\sqrt{4 - t^2}}{2}\, \frac{\sqrt{2}}{2 - t} \leqslant \frac{1}{3} + \frac{\sqrt{2}\sqrt{6}}{2}$$

for every $0 \leqslant t \leqslant 1$. Summarizing, we have $\omega(D[p_0(t)]) \leqslant 3$. ∎

We have arrived at the last step of our proof.

LEMMA 4. *Let E be a Banach space and ω be a real-analytic (resp., C^p, $p = 1, 2, ..., \infty$) norm on E. Let $z \in S_\omega = \{x \in E \mid \omega(x) = 1\}$, and let P_z be the hyperplane tangent to S at $-z$. Then the stereographic projection $\pi_z\colon S_\omega \backslash \{z\} \xrightarrow{\text{onto}} P_z$ establishes a real-analytic (resp., C^p) isomorphism.*

Proof. Let $x^* \in E^*$ be the derivative of ω at z. Then $P_z = \{x \in E \mid x^*(x + z) = 0\}$. We have

$$\pi_z(x) = z + \frac{2x^*(z)}{x^*(z - x)}(x - z), \qquad x \in S_\omega \backslash \{z\}.$$

To obtain π_z^{-1}, let $y \in P_z$ and find the unique t such that $\omega(z + t(y - z)) = 1$. We have that $\pi_z^{-1}(y) = z + t(y - z)$. Since for $F(y, t) = \omega(z + t(y - z)) - 1$ we have

$$\frac{\partial F(y, t)}{\partial t} = D[\omega(z + t(y - x))](y - z) \neq 0.$$

The implicit function theorem implies that $y \mapsto t = t(y)$ is real-analytic (resp., of class C^p). The lemma easily follows. ∎

Proof of Theorem. We can identify H with the product $l^2 \times H'$, where H' is a Hilbert space. Let $T: l^2 \times H' \to l^2 \times H'$ be given by $T(x, v) = ((x_n/2^n), v)$, $(x, v) = ((x_n)_{n=0}^{\infty}, v) \in l^2 \times H'$. Consider the incomplete norm ω on $l^2 \times H'$ induced by the operator T. Apply Lemma 1 to the original norm $\omega_1 = \|\cdot\|$ and the incomplete, real-analytic one, $\omega_2 = \omega$ to obtain a real-analytic isomorphism $h_1: S \xrightarrow{\text{onto}} S_\omega$. Then apply Lemma 3 to find a path $p: [0, 1) \to \hat{S}_\omega \cap (l^2 \times \{0\}, \omega)$ satisfying the requirements (a)–(c) of Proposition 1. Then apply Proposition 1 to obtain a real-analytic isomorphism $h: S_\omega \backslash \{z\} \to S_\omega$ for some $z \in S_\omega$. Apply Lemma 4 to $l^2 \times H'$, ω and to the above z to get a real-analytic isomorphism of $S_\omega \backslash \{z\}$ onto P_z. Let $i: P_z \to H$ be the composition of the affine map $x \to x + z$ of P_z onto a closed 1-codimensional subspace H_0 of H, and an isomorphism of H_0 onto H. Finally, set $f = i \circ \pi_z \circ h^{-1} \circ h_1: S \to H$. ∎

Remark 3. In case of $H = l^2$ we have the following formulas for h_1, h, i and π_z, the components of f,

$$h_1(x) = \frac{x}{\sqrt{\sum_{n=0}^{\infty} (x_n/2^n)^2}}, \qquad x = (x_n)_{n=0}^{\infty} \in S;$$

$$h(x) = \left(x_n - 2 \sum_{n=0}^{\infty} (x_n d^{2n}(x)/2^{2n} \right)_{n=0}^{\infty}, \qquad x = (x_n)_{n=0}^{\infty} \in S_\omega \backslash \{(1, 0, \ldots)\},$$

where $d(x) = \frac{1}{24}\sqrt{(x_0 - 1)^2 + \sum_{n=1}^{\infty} (x_n/2^n)^2}$;

$$\pi_z(x) = \left(-1, \frac{2x_1}{1 - x_0}, \frac{2x_2}{1 - x_0}, \ldots \right), \qquad x = (x_n)_{n=0}^{\infty} \in S_\omega \backslash \{(1, 0, \ldots)\};$$

$$i(x) = (x_1, x_2, \ldots), \qquad x = (-1, x_1, x_2, \ldots) \in l^2.$$

Here, $S_\omega = \{(x_n)_{n=0}^{\infty} \in l^2 \mid \sum_{n=0}^{\infty} (x_n/2^n)^2 = 1\}$.

2. A Generalization to Banach Spaces

Let E be a Banach space that admits a continuous injective operator $T\colon E \to H$ into a Hilbert space H such that T^{-1} is not continuous. Moreover, let $(F, \|\cdot\|_F)$ be a Banach space whose norm $\|\cdot\|_F$ is real-analytic (resp., of class C^p, $p = 1, 2, ..., \infty$). We will consider the space $Z = E \times F$ and the incomplete norm $\omega(x, y) = \sqrt{\|T(x)\|^2 + \|y\|_F^2}$. We see that ω is a real-analytic (resp., of class C^p).

Proposition 2. *There exists a real-analytic (resp., C^p) isomorphism $h\colon S_\omega\backslash\{z_0\} \xrightarrow{\text{onto}} S_\omega$, where $S_\omega = \{z \in Z \mid \omega(z) = 1\}$ and z_0 is some point in S_ω.*

Proof. We shall show the existence of a path p that satisfies (a)–(c) of Proposition 1. Then, making suitable adjustments, we shall follow the proof of Proposition 1.

Since $E \times \{0\} \subset Z$ is ω-incomplete, we can pick $\hat{z} \in \hat{Z}_\omega \backslash Z$ which is in $\hat{E}_\omega \times \{0\}$, the ω-closure of $E \times \{0\}$ in $\hat{Z}_\omega$. We can assume that $\omega(\hat{z}) = 1$. By a result of [Dob 1, Sublemma 3.2] there exists a path $q\colon [0, 1) \to \hat{Z}_\omega$ such that

(a′) $q(0) = \hat{x}$ and $q \mid (0, 1) : (0, 1) \to E$;

(b′) $q \mid (0, 1)$ is real-analytic as a map into E;

(c′) $\omega(q(t) - q(s)) \leqslant M\,|t - s|$ for some $M > 0$ and all t, s.

Since $\hat{x} \in S_\omega$, there exists $t_0 > 0$ so that

(d) $\frac{1}{2} \leqslant \omega(q(t)) \leqslant 2$ if $0 \leqslant t \leqslant t_0$.

Let $p(t) = q(t)/\omega(q(t))$. Clearly, such p fulfills the items (a) and (b) of Proposition 1 (note that ω restricted to $E \times \{0\}$ is real-analytic). To show (c), we only need to check that the derivative of $p(t)$, as a map into $\hat{Z}_\omega$, is bounded. We have

$$D[p(t)](v) = \frac{1}{\omega^2(q(t))} D[\omega(q(t))](v) \cdot q(t) + \frac{1}{\omega(q(t))} D[q(t)].$$

From (c′) and (d), it follows that the second summand is ω-bounded on $[0, t_0]$. Since $D[\omega(q(t))](v) = \langle D[q(t)](v), q(t)\rangle_\omega/\omega(q(t))$, using one more time (c′) and (d), the first summand of $D[p(t)](v)$ is also ω-bounded on $[0, t_0]$.

Pick $z_0 \in S_\omega \cap (E \times \{0\})$ so that $\langle z_0, \hat{x}\rangle_\omega \neq 0$ and let $d(z) = L\omega(z - z_0)$, $z \in S_\omega$, for a suitable constant $L > 0$. For $z = (x, y) \in \hat{S}_\omega$, we let

$$h(z) = (A(p(d(z))(x), (y),$$

where A is that of Lemma 2 (here, $\langle x, x' \rangle_\omega = \langle Tx, Tx' \rangle$, where $\langle \cdot, \cdot \rangle$ is the inner product on H). For $x_0 \in \hat{S}_\omega \cap \hat{E}_\omega \times \{0\}$, we write

$$\bar{A}(z) = (A(x_0)(x), y), \qquad z = (x, y) \in \hat{Z}_\omega.$$

Employing Lemma 2(a), we see that

$$\begin{aligned} \omega^2(\bar{A}(z)) &= \omega^2(A(x_0)(x)) + \|y\|_F^2 \\ &= \|x\|^2 + \|y\|_F^2. \end{aligned}$$

It follows that $\bar{A}: \hat{Z}_\omega \to \hat{Z}_\omega$ is an isometry. Moreover, since $\bar{A}^2(z) = (A^2(x_0), y) = (x, y)$ (use Lemma 2(a)), then $\bar{A}^2$ is the identity operator.

Remark 4. The observations from Remark 1 hold true in the case of $\hat{Z}_\omega$.

COROLLARY 1. *Let E be an infinite-dimensional separable Banach space (or, more generally admitting a total sequence $\{x_n^*: n \in \mathbb{N}\} \subset E^*$). If E admits a real-analytic (resp., C^p, $p = 1, 2, ..., \infty$) norm $\|\cdot\|$, then the unit $\|\cdot\|$-sphere $S = \{x \in E \mid \|x\| = 1\}$ is real-analytically (resp., C^p) isomorphic to a 1-codimensional closed linear subspace E_0 of E.*

Proof. We can assume that $\|x_n^*\| \leqslant 1$, $n \in \mathbb{N}$. Let $\omega(x) = \sqrt{\sum_{n=0}^{\infty} (x_n^*(x)/2^n)^2}$, $x \in E$. We can further assume that the norm ω is incomplete on E. (Otherwise, (E, ω) would be a separable, infinite dimensional, Hilbert space and we could easily find an incomplete continuous norm ω' on (E, ω); we could then replace ω by ω'.)

By Lemma 1, there exists a real-analytic (resp., C^p) isomorphism $h_1: S \xrightarrow{\text{onto}} S_\omega = \{x \in E \mid \omega(x) = 1\}$. By a special case of Proposition 2 (where $F = \{0\}$), there exists $x_0 \in S_\omega$ and a real-analytic (resp., C^p) isomorphism $h: S_\omega \backslash \{x_0\} \xrightarrow{\text{onto}} S_\omega$. Let $\pi_{x_0}: S_\omega \backslash \{x_0\} \to P_{x_0}$ be the stereographic projection of Lemma 4, and $a(x) = x + x_0$, $x \in P_{x_0}$. We see that $f = a \circ \pi_{x_0} \circ h^{-1} \circ h_1$ is a required isomorphism of S onto $P_{x_0} + x_0$; the latter space, in turn, is isomorphic to E_0. ∎

Remark 5. Observe that if E' is a dense linear subspace of E, then we can always pick x_0 in the ω-closure of E'. An inspection of the proof of Corollary 1 yields that f is a real-analytic isomorphism of $S \cap E'$ onto a 1-codimensional closed linear subspace of E'. In particular, the unit sphere of any infinite-dimensional pre-Hilbert space H is real-analytically isomorphic to a closed 1-codimensional subspace H_0 (which itself may not be isomorphic to H, see [vM]).

580/134/2-8

COROLLARY 2. *Let μ be a measure so that $L^p(\mu)$ is infinite-dimensional. If $p=2n$, $n\in\mathbb{N}$, then the unit sphere of $L^{2n}(\mu)$ (with respect to the standard norm $\|\cdot\|_{2n}$) is real-analytically isomorphic to $L^{2n}(\mu)$.*

Proof. It can be checked that $x \to \|x\|_{2n}^{2n}$ is a polynomial [DGZ, p. 184], hence $\|\cdot\|_{2n}$ is real-analytic. Moreover, $L^p(\mu)$ can always be represented as $E\times F$ for some separable Banach space. (Choose a sequence of measurable sets (A_n) with $0<\mu_n(A_n)<\infty$ with $A_i\cap A_j=\varnothing$ for $i\neq j$, and let E be the closed span of characteristic functions of A_n's. Then, E is isomorphic to l^p, and E is complemented in $L^p(\mu)$. This also shows that a closed 1-codimensional subspace of $L^p(\mu)$ is isomorphic to $L^p(\mu)$.)

Now we can repeat the argument of the proof of Corollary 1 (this time we use the full strength of Proposition 2). ∎

4. Final Remarks

Let E be an infinite-dimensional separable Banach space whose norm $\|\cdot\|$ is real-analytic (resp., of class C^p, $p=1, 2, ..., \infty$). Then employing results of [Dob1] and Corollary 1, we get

COROLLARY 3. *For every compact set $K\subset S=\{x\in E \mid \|x\|=1\}$, there exists a real-analytic (resp., C^p) isomorphism of $S\backslash K$ onto S.*

In the case of $p=1, 2, ..., \infty$ we can do better.

PROPOSITION 3. *There exists a C^p isomorphism $f\colon E\backslash K \xrightarrow{\text{onto}} E$ such that $f(\lambda S\cap(E\backslash K))=\lambda S$ for every $\lambda\geqslant 0$. Moreover, f can be chosen to have support in an arbitrary neighborhood G of S (i.e., $f(x)=x$ off G).*

LEMMA 5. *Let ω be a C^p (resp., real-analytic) norm on E. There exists $x_0\in S_\omega=\{x\in E \mid \omega(x)=1\}$, $\varepsilon>0$, and a C^p (resp., real-analytic) isomorphism $h\colon U_\varepsilon\backslash I_{\varepsilon/2} \xrightarrow{\text{onto}} U_\varepsilon$ such that $h(\lambda S_\omega\cap E\backslash\{x_0\})=\lambda S_\omega$ for every $|\lambda-1|<\varepsilon$; here $U_\varepsilon=\{x\in E \mid |\omega(x)-1|<\varepsilon\}$ and $I_{\varepsilon/2}=\{tx_0 \mid |t-1|\leqslant\varepsilon/2\}$.*

Proof. As indicated in the proof of Proposition 1 (see also the proof of Proposition 2), the formula describing h establishes a C^p (resp., real-analytic isomorphism of $U_\varepsilon\backslash\{x_0\}$ onto U_ε for some $\varepsilon>0$ and $x_0\in S_\omega$, and satisfies the condition that $h(\lambda S_\omega\cap E\backslash\{x_0\})=\lambda S_\omega$ for every $|\lambda-1|<\varepsilon$. We use here the fact that $\langle x_0, p(0)\rangle_\omega\neq 0$. However, if ε is small, then $\langle x, p(0)\rangle_\omega\neq 0$ for all $x\in I_{\varepsilon/2}$. Consequently, h will "delete" $x\in I_{\varepsilon/2}$ if only we guarantee that $d(x)=0$. Therefore, the function $d(x)=L\omega(x-x_0)$ must be replaced by ψ which is of class C^p (resp. real-analytic), vanishes precisely on the set $I_{\varepsilon/2}$ and satisfies $|\psi(x)-\psi(x')|\leqslant L\omega(x-x')$,

$x, x' \in U_\varepsilon$. As shown in [Dob1, Lemma 2.2] such real-analytic functions ψ always exist (provided E is separable). Hence, replacing d by a suitable ψ we see that h is a required isomorphism. ∎

LEMMA 6. *Using the notation of Lemma 5, there exists a C^p (resp., real-analytic) isomorphism h_2 of $U_{\varepsilon/4}$ onto $P_{x_0} \oplus [-1-(\varepsilon/4), -1+(\varepsilon/4)] \cdot x_0 \subset E$, so that $h_2(\lambda S_\omega) = P_{x_0} \oplus \{-\lambda x_0\}$ for every $|\lambda - 1| < \varepsilon/4$, where P_{x_0} is the hyperplane tangent to S_ω at $-x_0$.*

Proof. It is clear that since P_x is tangent to S_ω at $-x_0$, then $P_{x_0} \oplus \{-\lambda x_0\}$ is tangent to λS_ω at $-\lambda x_\omega$ for every $|\lambda - 1| < \varepsilon/4$. Let π_λ be the stereographic projection of $\lambda S_\omega \backslash \{\lambda x_0\}$ onto $P_{x_0} \oplus \{-\lambda x_0\}$. We let $h_2(x) = \pi_\lambda \circ h^{-1}(x)$, $x \in U_{\varepsilon/4}$, where h is that of Lemma 5. ∎

Proof of Proposition 3. First we show that given a compact subset L of S_ω, there exists a C^p (resp., real-analytic) isomorphism H of $U_{\varepsilon/4} \backslash L$ onto $U_{\varepsilon/4}$ such that $H(\lambda S_\omega \cap (E \backslash L)) = \lambda S_\omega$ for $|\lambda - 1| < \varepsilon/4$ and such that $H(x) = x$ if $|\omega(x) - 1| > \varepsilon/8$.

To this end, we use [Dob1, Corollary 6.4] to find λ-level preserving C^p isomorphism h_3 of $(P_{x_0} \oplus \mathbb{R}x_0) \backslash h_2(L)$ onto $P_{x_0} \oplus \mathbb{R}x_0$ so that $h_3(z) = z$ for every $z = (x, \lambda x_0)$, $|\lambda + 1| > \varepsilon/8$. We let $H = h_2^{-1} \circ h_3 \circ (h_2 \mid S \backslash L)$.

To finish the proof, let h_1 be the map of Lemma 1 defined by the same formula on $G_{\varepsilon/4} = \{x \in E \mid |\|x\| - 1| < \varepsilon/4\}$ for $\omega_1 = \|\cdot\|$ and $\omega_2 = \omega$. Construct H as above to delete $L = h_1(K)$, and let

$$f(x) = \begin{cases} h_1^{-1} H h_1(x), & x \in G_{\varepsilon/4} \\ x, & |\|x\| - 1| > \varepsilon/8. \end{cases}$$

It is clear that f has the required properties (obviously, $\varepsilon > 0$ can be chosen as small as one wishes). ∎

REFERENCES

[Bes] C. BESSAGA, Every infinite-dimensional Hilbert space is diffeomorphic with its unit sphere, *Bull. Acad. Polon. Sci. Sér. Sci. Math.* **14** (1966), 27–31.

[BK] D. BURGHELEA AND N. H. KUIPER, Hilbert manifolds, *Ann. of Math.* **90** (1969), 379–417.

[DGZ] R. DEVILLE, G. GODEFROY, AND V. ZIZLER, "Smoothness and Renormings in Banach Spaces," Longman Scientific & Technical, Essex/New York, 1993.

[Die] J. DIEUDONNÉ, "Foundations of Modern Analysis," Academic Press, New York/London, 1960.

[Dob1] T. DOBROWOLSKI, Smooth and R-analytic negligibility of subsets and extension of homeomorphism in Banach spaces, *Studia Math.* **65** (1979), 115–139.

[Dob2] T. DOBROWOLSKI, Extension of Bessaga's negligibility technique to certain infinite-dimensional groups, *Bull. Acad. Polon. Sci. Sér. Sci. Math.* **26** (1978), 535–545.

[EE] J. EELS AND K. B. ELWORTHY, Open embeddings of certain Banach manifolds, *Ann. of Math.* **91** (1970), 465–485.

[Lang] S. LANG, "Differential Manifolds," Springer-Verlag, New York/Berlin/Heidelberg/Tokyo, 1985.

[vM] J. VAN MILL, Domain invariance in infinite-dimensional linear spaces, *Proc. Amer. Math. Soc.* **101** (1987), 173–180.

[Wh] E. F. WHITTLESEY, Analytic functions in Banach spaces, *Proc. Amer. Math. Soc.* **16** (1965), 1077–1083.