Projections of a Regular Polyhedron

Paul Barbaroux

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Il this note we focus on the following exercise[1] :

Exercise 1 : *In the usual 3-dimensional euclidian space, consider a regular polyhedron and a plane $\mathcal{P}$. We denote S the sum of squared lengths of the edges, and $S_{\mathcal{P}}$ the sum of squared lengths of the orthogonal projections of the edges on $\mathcal{P}$. Prove that $S_{\mathcal{P}}/S = 2/3$. Then generalize.*

Before we give a solution, let's begin with a simpler statement which will allow to understand the main idea : *sphericity of an ellispoïd of inertia can be viewed as the geometric form of Schur's Lemma.*

Exercise 2 : *In the usual 3-dimensional euclidian space, let $\mathcal{S}$ be the set of vertices of a regular polyhedron, and $\mathcal{D}$ a line passing through the center. Show that $\sum_{s\in\mathcal{S}} d(s,\mathcal{D})^2$ does not depend on $\mathcal{D}$.*

A physicist would give a one-line solution by noticing that $\sum_{s\in S} d(s,\mathcal{D})^2$ is the moment of inertia of S with respect to $\mathcal{D}$ (after providing each vertex with a point mass of value 1), then arguing that the ellipsoïd of inertia is a sphere « from reasons of symmetry ». We will first formalize correctly the symmetry argument, that is in an intrinsic way (without use of a coordinates system).

We will replace points by vectors by setting an origin at the center of the polyhedron. In this note the ambiant space will be a euclidian vector space[2] E of dimension $n \neq 0$. The inner product will be denoted by $(\cdot \mid \cdot)$. Given a vector subspace F, p_F denotes the orthogonal projection on F. Given a set $A \subset E$, we say that an isometry g is an ***isometry of*** A if the set A is globally invariant under g. The set of isometries of A is a subgroup of $O(E)$.

First, the symmetry argument relies on the fact that « no direction is particular ». Mathematically, this leads to the following definition : we say that a set A of vectors of a euclidian space ***has enough symmetries*** if the group of isometries of A is irreducible[3].

It turns out that this is actually the case for a regular polyhedron :

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(1) Readers are invited to look for a solution, based on concepts rather than computations.

(2) Real finite-dimensional vector space endowed with an inner product.

(3) We recall that a subgroup G of $\mathrm{GL}(E)$ is ***irreducible*** if the only subspaces of E stable under the action of G are trivial.

Proposition : *In a 3-dimensional euclidian space, the set $\mathcal{S}$ of vertices of a regular polyhedron has enough symmetries.*

Proof : The group G of isometries of $\mathcal{S}$ contains at least two rotations of distinct axis and angles not multiple of π (consider rotations stabilizing two adjacent faces). Let F be a G-stable subspace of E, $F \neq \{0\}$. There exists a vector $a \in F \setminus \{0\}$. The axis of at least one of the rotations, call it r, does not contain a. Hence a is not invariant by r, neither anti-invariant (since $\dim(E) = 3$ the only rotations with an anti-invariant vector $a \neq 0$ have angle π). Hence a and $r(a)$ are linearly independant and $\dim F \geqslant 2$. But $F^\perp$ is also G-stable, hence $\dim F^\perp \geqslant 2$ or $F^\perp = \{0\}$. Since $\dim E = 3$ the only possibility is $F^\perp = \{0\}$ that is $F = E$, hence G is irreducible.

We can observe that if F is a subspace of E and $x \in E$ then $d(x, F) = ||p_{F^\perp(x)}||$. When $\dim E = 3$ the orthogonal of a line is a plane, thus statement of ex. 2 is a special case of the following theorem, which additionally gives the value $\sum_{s\in\mathcal{S}} d(s, \mathcal{D})^2 = (2/3) \times \sum_{s\in\mathcal{S}} ||s||^2$.

Theorem 1 : *Let E be a euclidian vector space and $A \subset E$ a finite set. If A has enough symmetries, then for any subspace $F \subset E$ one has*

$$\sum_{a\in A} ||p_F(a)||^2 = \frac{\dim F}{\dim E} \sum_{a\in A} ||a||^2 \,.$$

Proof : For $x \in E$ and $A \subset E$ a finite set, let

$$u_A(x) = \sum_{a\in A} (x \mid a)\, a \,.$$

u_A is clearly a self-adjoint operator[(1)], hence has an eigenvalue λ. But u_A commutes with all isometries of A : if g is such an isometry, then

$$g(u_A(x)) = \sum_{a\in A} (x \mid a) g(a) = \sum_{a\in A} (g(x) \mid g(a)) g(a) = \sum_{b\in A} (g(x) \mid b) b = u_A(g(x)) \,.$$

Therefore, G stabilizes the eigenspace $\mathrm{Ker}(u_A - \lambda\,\mathrm{id})$. But G is irreducible, hence $\mathrm{Ker}(u_A - \lambda\,\mathrm{id}) = E$ that is $u_A = \lambda\,\mathrm{id}$. [(2)]

Now for every $e \in E$ such that $||e|| = 1$, one has

$$\sum_{a\in A} (e \mid a)^2 = \sum_{a\in A} (e \mid a)(a \mid e) = (u_A(e) \mid e) = (\lambda e \mid e) = \lambda ||e||^2 = \lambda \,.$$

By denoting $(e_i)_{1\leqslant i\leqslant p}$ an orthonormal basis of the subspace F, we obtain

$$\sum_{a\in A} ||p_F(a)||^2 = \sum_{a\in A} \sum_{i=1}^{p} (e_i \mid a)^2 = \sum_{i=1}^{p} \sum_{a\in A} (e_i \mid a)^2 = \sum_{i=1}^{p} \lambda = \lambda \dim F \,.$$

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(1) Actually u_A is the "dual operator of inertia" of A : if $||x|| = 1$ then $(u_A(x) \mid x)$ is the moment of inertia of A w.r.t. to the hyperplane $x^\perp$.

(2) Which can be viewed as Schur's Lemma applied to G and u_A.

The special case $F = E$ then gives

$$\sum_{a\in A} ||a||^2 = \lambda n \,,$$

which gives the value of λ and completes the proof.

Th. 1 can now be easily extended to a more general result that allows to treat not only ex. 2 but also ex. 1 as well as several other geometric situations. We just have to replace the use of a finite *set* of vectors by the one of a finite *family*. Let I be a set (of "indices"). We suppose given an action $(g, i) \mapsto g \cdot i$ of the orthogonal group $O(E)$ on I. We say that a family of vectors $(x_i)_{i\in I}$ is ***admissible*** if $g(x_i) = x_{g\cdot i}$ for every $i \in I$ and $g \in O(E)$. As before an isometry of a set $J \subset I$ is an isometry that leaves J globalement unchanged, and we say that J has enough symmetries if its group of isometries is irreducible. Then we have

Theorem 2 : *Let $f = (x_i)_{i\in I}$ be an admissible family of vectors in a euclidian vector space E and $J \subset I$ a finite set having enough symmetries. Then for every subspace $F \subset E$ one has*

$$\sum_{i\in J} ||p_F(x_i)||^2 = \frac{\dim F}{\dim E} \sum_{i\in J} ||x_i||^2 \,.$$

The proof works in the same way as for th. 1 by now considering $u_f(x) = \sum_{i\in J} (x \mid x_i)\, x_i$ instead of $u_A(x) = \sum_{a\in A} (x \mid a)\, a$.

Remarks : *a*) Th. 2 gives back th. 1, when applied to $I = E$ and the identity-indexed family.

b) Ex. 2 is solved by taking

- $I = E \times E$ and, for $(a, b) \in I$, and $g \in O(E)$, $g \cdot (a, b) = (g(a), g(b))$.
- $x_{(a,b)} = b - a$. The admissibility of $(x_{(a,b)})$ comes from g's linearity.
- $J = \{(a, b) \in E \times E \,,\ \{a, b\}$ is an edge of the polyhedron$\}$ (hence two ordered pairs (a, b) and (b, a) are given for each edge $\{a, b\}$ such that $a \neq b$). Obviously J has enough symmetries, since any isometry of the polyhedron sends every edge to an edge, hence is an isometry of J.

c) A polyhedron does not need to be regular in order to ex. 1 to hold : it just has to have the *same isometry group as the set vertices of a regular polyhedron*[(1)]. One can obtain such polyhedra by truncating regular polyhedra (for instance, conclusion of ex. 1 remains true for *archimedean solids*[(2)]) or, on the opposite, by having other polyhedra "grow" on

(1) It is even enough for its isometry group to *contain* the isometry group of some regular polyhedron.

(2) See *e.g.*

`https://en.wikipedia.org/wiki/Archimedean_solid`

`https://mathcurve.com//polyedres/archimedien/archimedien.shtml` faces. For instance, one can “glue” on each face of a regular polyhedron a pyramid or a prism of given height.

d) Let’s call a ***skeleton*** any finite set $J \subset E \times E$. Starting with skeletons of edges of regular polyhedra, or given by remark *c*), one can easily construct many other ones satisfying the statement of ex. 1 by noticing that the set of such skeletons is stable under translations and disjoint unions.