A PRIMER ON GALOIS CONNECTIONS

M. Erné, J. Koslowski, A. Melton, G. E. Strecker

ABSTRACT: We provide the rudiments of the theory of Galois connections (or residuation theory, as it is sometimes called) together with many examples and applications. Galois connections occur in profusion and are well-known to most mathematicians who deal with order theory; they seem to be less known to topologists. However, because of their ubiquity and simplicity, they (like equivalence relations) can be used as an effective research tool throughout mathematics and related areas. If one recognizes that a Galois connection is involved in a phenomenon that may be relatively complex, then many aspects of that phenomenon immediately become clear; and thus, the whole situation typically becomes much easier to understand.

KEY WORDS: Galois connection, closure operation, interior operation, polarity, axiality

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INTRODUCTION

Mathematicians are familiar with the following situation: there are two "worlds" and two transforming functions back and forth between these worlds. Moreover, after an object has been transformed from one world to the other and then back to the first world, a certain stability is reached in that further transformations produce the same results. In particular, three transformations always yield the same result as just one (no matter where one starts). Further, both worlds often carry natural orders. When the transforming processes respect these orders, then we frequently have situations that are both simple and rich and that can be handled in a very elegant way. Their simplicity is reflected in the succinct definition (see Definition 1), and their richness is manifested in the wealth of results that follow from the definition (see Propositions 2–6). These pleasant situations are called *Galois connections* after Evariste Galois [1], whose work initiated the study of the connection between the world of intermediate fields of a field extension $E:F$ and the world of subgroups of the group of automorphisms of E that fix the subfield F (cf. Examples 1 and 19). Today this area is known as *Galois theory.*

Since the proofs of many results in this paper are either well-known or easily obtained, we do not include them. Galois connections were originally expressed in a symmetric but contravariant form with transformations that reverse (rather than preserve) order. Early references to this form are [8], [22], [44], and [45]. We use the covariant form since it is more convenient, e.g., compositions of Galois connections are handled more easily; it allows for more natural categorical explanations (e.g., by means of adjunctions); and it is more applicable to computer science situations (where relative information preservation is important). For references to the covariant form see [7], [51], [9], [32], [25], [35], [15], [42], [31], and [19].

[1] Galois' main results were published fourteen years after his early death (at the age of 21 in a duel) by Liouville in his *Journal de mathématiques pures et appliquées* (1846). For a translation of Galois' original notes *Memoire sur les conditions de résolubilité des équations par radicaux*, see the text by Edwards [14].

THE DEFINITION AND SOME OF ITS CONSEQUENCES

We formulate all our results in terms of partially ordered sets (or *posets*), i.e., sets equipped with a reflexive, transitive, and antisymmetric relation. Everything can easily be generalized to pre-ordered sets (i.e., one may drop the antisymmetry requirement) and even to pre-ordered classes. Applications of these generalizations can be found in Examples 10, 22, and 23, and in [10]. For a poset $\mathscr{P} = \langle P, \leq \rangle$, the order-theoretical dual $\langle P, \geq \rangle$ is denoted by $\mathscr{P}^{\mathrm{op}}$.

DEFINITION 1: Consider posets $\mathscr{P} = \langle P, \leq \rangle$ and $\mathscr{Q} = \langle Q, \sqsubseteq \rangle$. If $P \xrightarrow{\pi_*} Q$ and $Q \xrightarrow{\pi^*} P$ are functions such that for all $p \in P$ and all $q \in Q$

$$p \leq \pi^*(q) \qquad \text{iff} \qquad \pi_*(p) \sqsubseteq q \tag{1}$$

then the quadruple $\pi = \langle \mathscr{P}, \pi_*, \pi^*, \mathscr{Q} \rangle$ is called a **Galois connection**.

We also write $\mathscr{P} \xrightarrow{\pi}\!\!\bullet\, \mathscr{Q}$ (or sometimes just $\langle \pi_*, \pi^* \rangle$) for the whole Galois connection. π_* and π^* are called the **coadjoint part** and the **adjoint part** of π, respectively. We write P^π for the π^*-image of Q, and $\mathscr{P}^\pi$ for $\langle P^\pi, \leq \rangle$. Similarly, Q_π denotes the π_*-image of P, and $\mathscr{Q}_\pi$ stands for $\langle Q_\pi, \sqsubseteq \rangle$. The elements of P^π (resp. Q_π) are called **π-closed** (resp. **π-open**). We say that π is a **coreflection**[2] (resp. **reflection**) if π_* (resp. π^*) is a one-to-one function, and an **interior** (resp. **closure**) **connection** if π_* (resp. π^*) is an inclusion. π is called an **isomorphism** if both π_* and π^* are one-to-one, and hence mutually inverse bijections (see Proposition 3(5) and (9) below). We call a function between posets (**co**)**adjoint** iff it is the (co)adjoint part of a Galois connection.

DEFINITION 2: (1) A self-map f of a poset $\langle P, \leq \rangle$ is said to be **increasing** (resp. **decreasing**) if $p \leq f(p)$ (resp. $f(p) \leq p$) for each $p \in P$. A **closure** (resp. **interior**) **operation** on $\langle P, \leq \rangle$ is an order-preserving, idempotent, and increasing (resp. decreasing) self-map.

(2) A **closure system** of $\langle P, \leq \rangle$ is a subset Q of P such that for each $p \in P$ there is a smallest $q \in Q$ with $p \leq q$, called the **closure** of p and denoted by p^-. The order-theoretically dual notions are **interior system** and **interior** p° of p. By a closure system **on** a set X, we mean a closure system of the **power set lattice** $\boldsymbol{P}(X) = \langle P(X), \subseteq \rangle$.

Closure connections π bijectively correspond to closure operations: assign to π the composite $\pi^* \circ \pi_*$ (cf. Propositions 3(4) and 6). Mapping each closure operation on a poset to its image yields a bijective correspondence with closure systems of the poset.

PROPOSITION 1: *Let Q be a closure (interior) system of $\mathscr{P}$.*

(1) *A subset A of Q has an infimum (supremum) in $\mathscr{Q} = \langle Q, \leq \rangle$ iff it has an infimum (supremum) in $\mathscr{P}$, and whenever either exists, they are equal.*

[2] In Computer Science coreflections are frequently called *embedding-projection pairs.*

(2) *If $B \subseteq Q$ has a supremum (an infimum) b in $\mathscr{P}$ then b^- (b°) is the supremum (infimum) of B in $\mathscr{Q}$.*

(3) *If $\mathscr{P}$ is a complete lattice, then the closure (interior) systems of $\mathscr{P}$ are those subsets that are closed under infima (suprema). Hence, under the induced order, closure (interior) systems of $\mathscr{P}$ are also complete lattices. They need not, however, be sublattices.* □

Galois connections behave well under composition and order-inversion.

PROPOSITION 2: *If $\mathscr{P} \xrightarrow{\pi}\bullet \mathscr{Q}$ and $\mathscr{Q} \xrightarrow{\rho}\bullet \mathscr{R}$ are Galois connections, so are $\mathscr{P} \xrightarrow{\rho\circ\pi}\bullet \mathscr{R}$, where $\rho\circ\pi = \langle \rho_* \circ \pi_*, \pi^* \circ \rho^* \rangle$, and $\langle Q, \sqsupseteq \rangle = \mathscr{Q}^{\mathrm{op}} \xrightarrow{\pi^{\mathrm{op}}}\bullet \mathscr{P}^{\mathrm{op}} = \langle P, \geq \rangle$, where $\pi^{\mathrm{op}} = \langle \pi^*, \pi_* \rangle$. In particular, the π-open elements of Q are precisely the π^{op}-closed ones.* □

PROPOSITION 3: *For any Galois connection $\mathscr{P} \xrightarrow{\pi}\bullet \mathscr{Q}$*

(1) *both π_* and π^* preserve order.*

(2) *π_* and π^* are mutual* **quasi-inverses**, *i.e., $\pi_* \circ \pi^* \circ \pi_* = \pi_*$ and $\pi^* \circ \pi_* \circ \pi^* = \pi^*$.*

(3) *$p \in P^\pi$ iff p is a fixed point of $\pi^* \circ \pi_*$, and $q \in Q_\pi$ iff q is a fixed point of $\pi_* \circ \pi^*$.*

(4) *$\pi^* \circ \pi_*$ is a closure operation on $\mathscr{P}$ that induces a closure connection $\mathscr{P} \xrightarrow{\hat{\pi}}\bullet \mathscr{P}^\pi$; similarly, $\pi_* \circ \pi^*$ is an interior operation on $\mathscr{Q}$ that induces an interior connection $\mathscr{Q}_\pi \xrightarrow{\check{\pi}}\bullet \mathscr{Q}$. Hence P^π is a closure system of $\mathscr{P}$, and Q_π is an interior system of $\mathscr{Q}$, so the results of Proposition 1 apply.*

(5) *$\mathscr{P}^\pi$ and $\mathscr{Q}_\pi$ are isomorphic posets; the restrictions of π_* and π^* to these posets yield an isomorphism $\mathscr{P}^\pi \xrightarrow{\tilde{\pi}}\bullet \mathscr{Q}_\pi$; and π canonically factors as the closure connection $\hat{\pi}$ followed by this isomorphism $\tilde{\pi}$ followed by the interior connection $\check{\pi}$; namely*

$$\mathscr{P} \xrightarrow{\pi}\bullet \mathscr{Q} \;=\; \mathscr{P} \xrightarrow{\hat{\pi}}\bullet \mathscr{P}^\pi \xrightarrow{\tilde{\pi}}\bullet \mathscr{Q}_\pi \xrightarrow{\check{\pi}}\bullet \mathscr{Q} \tag{2}$$

(6) *the functions π_* and π^* uniquely determine each other; in fact*

$$\pi_*(p) = \inf(\{\, q \in Q \mid p \leq \pi^*(q) \,\}) \qquad \text{and} \qquad \pi^*(q) = \sup(\{\, p \in P \mid \pi_*(p) \sqsubseteq q \,\})$$

This justifies calling π_* **the coadjoint** *(or* **lower adjoint***) of π^* and calling π^** **the adjoint** *(or* **upper adjoint***) of π_*.*

(7) *for each $p \in P^\pi$ its π^*-inverse image has $\pi_*(p)$ as the smallest element; similarly, the π_*-inverse image of each $q \in Q_\pi$ has $\pi^*(q)$ as the largest element.*

(8) *π_* preserves joins (i.e., suprema), and π^* preserves meets (i.e., infima).*

(9) *π is a reflection (i.e., π^* is one-to-one) iff π_* is surjective iff $\pi_* \circ \pi^* = id_Q$; and π is a coreflection (i.e., π_* is one-to-one) iff π^* is surjective iff $\pi^* \circ \pi_* = id_P$.* □

Now we see that Galois connections can be characterized by other simple conditions that sometimes are easier to verify than the definition above. Recall that $\uparrow p = \{\, x \in P \mid p \leq x \,\}$ is the **principal filter** generated by $p \in P$, and $\downarrow q = \{\, y \in Q \mid y \sqsubseteq q \,\}$ is the **principal ideal** generated by $q \in Q$. We write $P(Y) \xrightarrow{h^{\leftarrow}} P(X)$ for the inverse image function of $X \xrightarrow{h} Y$.

PROPOSITION 4: *Let* $\mathscr{P} = \langle P, \leq \rangle$ *and* $\mathscr{Q} = \langle Q, \sqsubseteq \rangle$ *be posets, and let* $P \xrightarrow{f} Q$ *and* $Q \xrightarrow{g} P$ *be functions.*

(1) *The following are equivalent:*

   (a) $\langle \mathscr{P}, f, g, \mathscr{Q} \rangle$ *is a Galois connection.*

   (b) f *and* g *preserve order,* $g \circ f$ *is increasing, and* $f \circ g$ *is decreasing.*

   (c) f *preserves order, and for each* $q \in Q$ *the largest element of* $f^{\leftarrow}(\downarrow q)$ *is* $g(q)$.

   (d) g *preserves order, and for each* $p \in P$ *the smallest element of* $g^{\leftarrow}(\uparrow p)$ *is* $f(p)$.

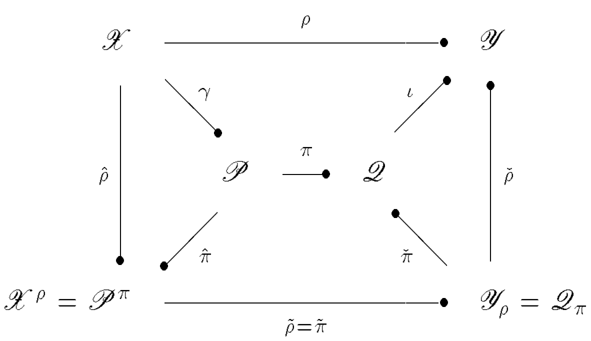
(2) f *is coadjoint iff* f *is* **residuated**, *i.e., iff* $f^{\leftarrow}$ *preserves principal ideals. If* $\mathscr{P}$ *is complete, this condition is equivalent to the preservation of arbitrary suprema by* f.

(3) g *is adjoint iff* g *is* **residual**, *i.e., iff* $g^{\leftarrow}$ *preserves principal filters. If* $\mathscr{Q}$ *is complete, this condition is equivalent to the preservation of arbitrary infima by* g. □

PROPOSITION 5: *By assigning to each coadjoint self-map of a poset* $\mathscr{P}$ *its adjoint, one obtains a dual isomorphism (reversing both the composition and the order) between the monoid of all coadjoint self-maps on* $\mathscr{P}$ *and the monoid of all adjoint self-maps on* $\mathscr{P}$*, both ordered pointwise. In particular, this isomorphism preserves idempotency. Moreover, coadjoints that are increasing (decreasing) correspond to adjoints that are decreasing (increasing).* □

Finally, let us have a look at the relativization of Galois connections.

PROPOSITION 6: *Let* P *be a closure system of a poset* $\mathscr{X} = \langle X, \leq \rangle$ *with closure connection* $\mathscr{X} \xrightarrow{\gamma}\!\!\bullet\; \mathscr{P} = \langle P, \leq \rangle$*, and let* Q *be an interior system of a poset* $\mathscr{Y} = \langle Y, \sqsubseteq \rangle$ *with interior connection* $\mathscr{Q} = \langle Q, \sqsubseteq \rangle \xrightarrow{\iota}\!\!\bullet\; \mathscr{Y}$. *Then every Galois connection* $\mathscr{X} \xrightarrow{\rho}\!\!\bullet\; \mathscr{Y}$ *with* $\rho_*(x) \in Q$ *for each* $x \in X$ *and* $\rho^*(y) \in P$ *for each* $y \in Y$ *induces a Galois connection* $\mathscr{P} \xrightarrow{\pi}\!\!\bullet\; \mathscr{Q}$ *such that the following diagram commutes*



(3)

In particular, ρ *and* π *induce the same isomorphism (and the lower trapezoid collapses iff* π *is already an isomorphism).*

Conversely, under the above hypothesis every Galois connection $\mathscr{P} \xrightarrow{\pi}\!\!\bullet\; \mathscr{Q}$ *is induced by a unique Galois connection* $\mathscr{X} \xrightarrow{\rho}\!\!\bullet\; \mathscr{Y}$*, namely the composite* $\mathscr{X} \xrightarrow{\gamma}\!\!\bullet\; \mathscr{P} \xrightarrow{\pi}\!\!\bullet\; \mathscr{Q} \xrightarrow{\iota}\!\!\bullet\; \mathscr{Y}$. *Hence the set of all Galois connections from* $\mathscr{P}$ *to* $\mathscr{Q}$ *with the pointwise order (on the coadjoints) is isomorphic to the poset of all Galois connections from* $\mathscr{X}$ *to* $\mathscr{Y}$ *that restrict to ones from* $\mathscr{P}$ *to* $\mathscr{Q}$.

EXAMPLES OF GALOIS CONNECTIONS

In each example of this section $\pi = \langle \mathscr{P}, \pi_*, \pi^*, \mathscr{Q} \rangle$ is a Galois connection. Many natural examples where $\mathscr{P}$ and $\mathscr{Q}$ are power sets are given in Sections 4 and 5.

EXAMPLE 1: (cf. [44], [3]) Let $E : F$ be a field extension, and $\mathscr{P}$ be the set of intermediate fields between F and E ordered by inclusion. Let $G = G\langle E, F \rangle$ be the group of all field automorphisms of E that fix F pointwise, and let $\mathscr{Q}$ be the set of all subgroups of G ordered by reverse inclusion. Define $\mathscr{P} \xrightarrow{\pi_*} \mathscr{Q}$ and $\mathscr{Q} \xrightarrow{\pi^*} \mathscr{P}$ by

$$\pi_*(L) = \{\, g \in G \mid g \text{ fixes } L \text{ pointwise} \,\}$$
$$\pi^*(H) = \{\, a \in E \mid \text{ every } h \in H \text{ fixes } a \,\}$$

This is the Galois connection that arises from Galois theory in its modern form, as presented, e.g., by Artin [3]. (Of course, neither modern algebraic notions such as field or automorphism group nor order-theoretical aspects occur explicitly in Galois' original work.) By known theorems of classical algebra, every *finite* subgroup H of G is π^{op}-closed, i.e., $\pi_*\pi^*(H) = H$, and $\pi^*(H) : F$ is a finite field extension with dimension $\boldsymbol{card}(H)$. Conversely, if $E : F$ is a finite extension, then $\boldsymbol{card}(G)$ is bounded by $\boldsymbol{dim}(E : F)$, and hence π is a reflection. If in addition $E : F$ is a **Galois extension**, i.e., if F is π-closed, then the fundamental theorem of Galois theory states that π is an isomorphism; in this case *every* intermediate field between F and E is π-closed. □

EXAMPLE 2: (cf. [57], [27]) For a function $X \xrightarrow{h} Y$ between two sets, let $\mathscr{P}$ and $\mathscr{Q}$ be the power sets of X and Y, respectively, both ordered by inclusion. Define the **direct image function** $\mathscr{P} \xrightarrow{h_\rightarrow} \mathscr{Q}$ and the **inverse image function** $\mathscr{Q} \xrightarrow{h^\leftarrow} \mathscr{P}$ by

$$h_\rightarrow(U) = \{\, y \in Y \mid \exists_{x \in U} \text{ with } h(x) = y \,\}, \qquad \text{the } \textbf{direct image} \text{ of } U \text{ under } h,$$
$$h^\leftarrow(V) = \{\, x \in X \mid \exists_{y \in V} \text{ with } h(x) = y \,\}, \qquad \text{the } \textbf{inverse image} \text{ of } V \text{ under } h.$$

Since $h_\rightarrow$ preserves unions and $\mathscr{P}$ is a complete lattice, by Proposition 4(2) $h_\rightarrow$ is the coadjoint part π_* of some Galois connection π. The corresponding adjoint π^* turns out to be $h^\leftarrow$, which by Proposition 3(8) must preserve intersections. The elements of $\mathscr{P}^\pi$ are called **h-saturated**. Notice that π^* also preserves unions and hence is a coadjoint.

If h is only a **partial function** $X \rightharpoonup Y$, i.e., h is defined on only a subset $X' \subseteq X$, then $h_\rightarrow$ still is coadjoint. Its adjoint π^* then maps $V \subseteq Y$ to the disjoint union of $h^\leftarrow(V)$ and $X - X'$. In particular, if $X' \neq X$, then π^* does not preserve the empty join (hence by Proposition 3(8) it cannot be coadjoint), and the inverse image function $h^\leftarrow$ does not preserve the empty meet (hence it is not adjoint). □

EXAMPLE 3: (cf. [27]) Even for a partial function $X \overset{h}{\rightharpoonup} Y$ the inverse image function $h^\leftarrow$ preserves unions, and hence by Proposition 4(2) is coadjoint. Let $\mathscr{P}$ and $\mathscr{Q}$ be the power sets of Y and X, respectively, both ordered by inclusion. (Note that the roles of X and Y are reversed as compared to Example 2.) Define $\mathscr{P} \xrightarrow{\pi_*} \mathscr{Q}$ to be $h^\leftarrow$. According to Proposition

3(6) the corresponding adjoint π^* must map $U \subseteq X$ to the union of all those subsets of Y whose inverse image under h is contained in U. It turns out that $\pi^*(U) = Y - h_\to(X - U)$.

Notice that $h_\to$ preserves unions but need not preserve intersections, while the adjoint to $h^\leftarrow$ we just described preserves intersections but need not preserve unions. □

EXAMPLE 4: (cf. [16]) Let h be a continuous map between topological spaces or, more generally, between closure spaces $\langle X, \mathscr{X}\rangle$ and $\langle Y, \mathscr{Y}\rangle$ (i.e., $\mathscr{X}$ and $\mathscr{Y}$ are closure systems on X and Y, respectively, and $h^\leftarrow$ maps $\mathscr{Y}$ into $\mathscr{X}$)[3]. The modified direct image function $\mathscr{X} \xrightarrow{h^-_\to} \mathscr{Y}$ maps $C \in \mathscr{X}$ to $(h_\to(C))^-$. Let $h^\leftarrow_-$ denote the restricted inverse image function from $\mathscr{Y}$ to $\mathscr{X}$. Then $\pi = \langle h^-_\to, h^\leftarrow_-\rangle$ is a Galois connection from $\langle \mathscr{X}, \subseteq\rangle$ to $\langle \mathscr{Y}, \subseteq\rangle$. In particular, $h^-_\to$ preserves joins (but not necessarily unions), and $h^\leftarrow_-$ preserves meets (i.e., intersections). $X \xrightarrow{k} Y$ is continuous from $\langle X, \mathscr{X}\rangle$ to $\langle Y, \mathscr{Y}\rangle$ iff there exists a coadjoint (= join-preserving) $\mathscr{X} \xrightarrow{K} \mathscr{Y}$ such that $K \circ \eta_X = \eta_Y \circ k$, where the "completion maps" $X \xrightarrow{\eta_X} \mathscr{X}$ and $Y \xrightarrow{\eta_Y} \mathscr{Y}$ send each point to its closure. Clearly, K then must agree with $k^-_\to$. This construction is fundamental for the representation of topological spaces by complete lattices and vice versa, and also for certain aspects in the theory of hyperspaces (see, e.g., [25] and [18]). □

EXAMPLE 5: Let $\mathscr{P}$ be the set of all open sets and $\mathscr{Q}$ that of all closed sets of a topological space with underlying set X, both ordered by inclusion. The topological closure operation $(\,)^-$ gives rise to a closure connection $\boldsymbol{P}(X) \xrightarrow{\gamma}\!\!\bullet\; \mathscr{Q}$, and the topological interior operation $(\,)^\circ$ induces an interior connection $\mathscr{P} \xrightarrow{\iota}\!\!\bullet\; \boldsymbol{P}(X)$. Their composite $\gamma \circ \iota$ is a Galois connection $\mathscr{P} \xrightarrow{\pi}\!\!\bullet\; \mathscr{Q}$ with $\pi_*(U) = U^-$ and $\pi^*(V) = V^\circ$. The π-closed members of $\mathscr{P}$ are the **regular open** (!) sets $U = U^{-\circ}$, and the π-open members of $\mathscr{Q}$ are the **regular closed** (!) sets $V = V^{\circ -}$. Thus $(\,)^-$ and $(\,)^\circ$ induce an isomorphism between the lattice of regular open sets and that of regular closed sets.

Composing π with the isomorphism $\mathscr{Q} \xrightarrow{(\,)'}\!\!\bullet\; \mathscr{P}^{\mathrm{op}}$ given by complementation yields a Galois connection $\mathscr{P} \xrightarrow{\pi'}\!\!\bullet\; \mathscr{P}^{\mathrm{op}}$ with $\pi'_*(U) = \pi'^*(U) = U^{-\prime} = U'^\circ$. Hence the lattice $\mathscr{P}^\pi = \mathscr{P}^{\pi'}$ of regular open sets is self-dual. In fact, these lattices are always Boolean (see Example 18).

Notice that the operation of taking the complement of the topological closure does not induce a Galois connection from $\boldsymbol{P}(X)$ to $\boldsymbol{P}(X)^{\mathrm{op}}$ (e.g., in the usual topology on the reals $\{0\}^{-\prime-\prime} = \emptyset \not\supseteq \{0\}$). Nevertheless the Galois connection π' can be used to prove the standard result that there are at most 14 sets that result from the application of complement and closure to any subset $U \subseteq X$ (since $U^{-\prime}$ is open, $U^{-\prime-\prime} = \pi'_*(U)$ is a fixed point of $\pi'_* \circ \pi'^*$). □

As can be seen from the following example, a similar phenomenon occurs with the compactness operator introduced in [13].

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[3] If $\mathscr{T}$ is a topology on X, and $\mathscr{X}$ consists of the $\mathscr{T}$-closed sets, then $\langle X, \mathscr{X}\rangle$ is a closure space.

EXAMPLE 6: Let $\mathscr{P}$ be the set of all topologies on X ordered by inclusion, and let $\mathscr{Q}$ be $\mathscr{P}^{\mathrm{op}}$. Consider the operator κ that maps any topology $\mathscr{T}$ on X to the topology obtained by using the $\mathscr{T}$-compact sets as a base for the closed sets. Even though κ preserves order from $\mathscr{P}$ to $\mathscr{Q}$ and vice versa (because κ *reverses* inclusion), for every infinite set the pair $\langle\kappa,\kappa\rangle$ is not a Galois connection (while for finite X it trivially is one). For example, if α is any ordinal, and $\mathscr{T}$ is the Alexandroff topology of all upper sets (cf. Example 9), then *every* subset of α is compact, and consequently, $\kappa^2(\mathscr{T})$ is the cofinite topology on α. For infinite α, this topology is neither coarser nor finer than $\mathscr{T}$. However, in all cases $\langle\kappa,\kappa\rangle$ induces a Galois connection between the κ-images of $\mathscr{Q}$ and $\mathscr{P}$. This follows immediately from the fact that κ preserves order and that $\kappa(\mathscr{T}) \subseteq \kappa^3(\mathscr{T})$ for any topology $\mathscr{T}$ (cf. [13], Theorem II (8) and (9)). Thus Proposition 3(2) gives an instant proof via Galois connections of Theorem II (10) of [13] that $\kappa^2 = \kappa^4$. □

EXAMPLE 7: Let $\mathscr{P}$ be the set of all topologies on Y, and let $\mathscr{Q}$ be the set of all topologies on X, both ordered by inclusion. Given a function $X \xrightarrow{h} Y$, define π_* and π^* by

$$\pi_*(\mathscr{S}) = \{\, h^{\leftarrow}(V) \mid V \in \mathscr{S} \,\} \qquad (\mathscr{S} \in \mathscr{P})$$
$$\pi^*(\mathscr{T}) = \{\, V \subseteq Y \mid h^{\leftarrow}(V) \in \mathscr{T} \,\} \qquad (\mathscr{T} \in \mathscr{Q})$$

Observing that h is a continuous function from $\langle X,\mathscr{T}\rangle$ to $\langle Y,\mathscr{S}\rangle$ iff $\pi_*(\mathscr{S}) \subseteq \mathscr{T}$ iff $\mathscr{S} \subseteq \pi^*(\mathscr{T})$, we see that π_* and π^* are in fact the coadjoint part and the adjoint part of a Galois connection $\mathscr{P} \xrightarrow{\pi}\!\bullet\, \mathscr{Q}$. Hence $\pi_*(\mathscr{S})$ is the coarsest topology $\mathscr{T}$ on X that makes h continuous from $\langle X,\mathscr{T}\rangle$ to $\langle Y,\mathscr{S}\rangle$ (called the **initial** topology with respect to h and $\mathscr{S}$), and $\pi^*(\mathscr{T})$ is the finest topology $\mathscr{S}$ on Y that makes h continuous from $\langle X,\mathscr{T}\rangle$ to $\langle Y,\mathscr{S}\rangle$ (called the **final** topology with respect to h and $\mathscr{T}$).

The π-closure of a topology $\mathscr{S}$ on Y is a new topology that consists of all those sets $B \subseteq Y$ with $h^{\leftarrow}(B) = h^{\leftarrow}(V)$ for some $V \in \mathscr{S}$. On the other hand, the interior operation $\pi_* \circ \pi^*$ maps a topology $\mathscr{T}$ on X to a new topology that consists of all those $U \in \mathscr{T}$ with $U = h^{\leftarrow}(B)$ for some $B \subseteq Y$.)

A topology $\mathscr{T}$ on X is π-open iff (each member of) $\mathscr{T}$ is h-**saturated** (see Example 2), and a topology $\mathscr{S}$ on Y is π-closed iff it is h-**discrete**, i.e., $h_{\rightarrow}(X)$ and all subsets of $Y - h_{\rightarrow}(X)$ are $\mathscr{S}$-open. By Proposition 3(5) the lattice of h-saturated topologies on X is isomorphic to the lattice of h-discrete topologies on Y. If h is one-to-one then every topology on X is h-saturated, and if h is onto then every topology on Y is h-discrete. □

EXAMPLE 8: (cf. [45]) Let $\boldsymbol{B}(X)$ and $\boldsymbol{F}(X)$ be the sets of all bases (for topologies) and of all filters on X, respectively. Both $\boldsymbol{B}(X)$ and $\boldsymbol{F}(X)$ are subsets of the double power set of X, where $\mathscr{B} \in \boldsymbol{B}(X)$ iff $U, V \in \mathscr{B}$ and $x \in U \cap V$ imply the existence of some $W \in \mathscr{B}$ with $x \in W \subseteq U \cap V$, and $\mathscr{F} \in \boldsymbol{F}(X)$ iff $X \in \mathscr{F}$ and $\mathscr{F}$ is closed under supersets and binary intersections. (To make $\boldsymbol{F}(X)$ into a complete lattice, we include the power set of X as a filter.) Define a Galois connection π from $\mathscr{P} = \langle \boldsymbol{B}(X), \subseteq\rangle$ to $\mathscr{Q} = \langle \boldsymbol{F}(X), \subseteq\rangle^X$, the set of all functions $X \longrightarrow \boldsymbol{F}(X)$ with the pointwise order induced by the inclusion on $\boldsymbol{F}(X)$, by

$$\pi_*(\mathscr{B})(x) = \{\, V \subseteq X \mid \exists_{B\in\mathscr{B}} \text{ with } x \in B \subseteq V \,\} \qquad \text{for all } x \in X$$
$$\pi^*(u) = \{\, U \subseteq X \mid \forall_{x\in U}\ U \in u(x) \,\}$$

The π-closed bases are precisely the topologies on X; in fact, $\pi^*\pi_*(\mathscr{B})$ is the topology generated by the base $\mathscr{B}$. The π-open filter functions $X \xrightarrow{u} \boldsymbol{F}(X)$ are the topological neighborhood functions, i.e., for each $V \in u(x)$ there exists a subset $U \subseteq V$ with $x \in U$ and $U \in u(y)$ for every $y \in U$. Thus we obtain the classical bijective correspondence between topologies and functions that assign neighborhood filters to points. $\square$

Example 9: For a topology $\mathscr{T}$ on X define a pre-order $\leq_{\mathscr{T}}$ on X by

$$x \leq_{\mathscr{T}} y \iff x \in U \text{ implies } y \in U \text{ for all } U \in \mathscr{T} \tag{4}$$

This condition is equivalent to $\{x\}^- \subseteq \{y\}^-$. The pre-order $\leq_{\mathscr{T}}$ is called the **specialization order** on X with respect to $\mathscr{T}$. It is a partial order iff $\mathscr{T}$ is a T_0-topology.

For a pre-ordered set $\langle X, \leq \rangle$ each $U \subseteq X$ generates an **upper set** $\uparrow U = \bigcup\{\uparrow x \mid x \in U\}$. Among the topologies with specialization $\leq$ there is a finest one, the **Alexandroff topology** $\mathscr{A}_\leq$, that consists of all upper sets $U = \uparrow U \subseteq X$ (cf. [1]), and a coarsest one, the **upper topology** $\mathscr{U}_\leq$, that is generated by the complements of the principal ideals $\downarrow x$, for $x \in X$ (cf. [25]). The closed sets with respect to $\mathscr{A}_\leq$ are precisely the open sets with respect to $\mathscr{A}_\geq$, i.e., the **lower sets** $\downarrow V = \bigcup\{\downarrow x \mid x \in V\}$.

Let $\mathscr{P}$ be the set of all topologies on X ordered by inclusion, and let $\mathscr{Q}$ be the set of all pre-orders on X ordered by reverse inclusion. One obtains a Galois connection $\mathscr{P} \xrightarrow{\pi}\!\!\bullet\ \mathscr{Q}$ via

$$\pi_*(\mathscr{T}) = \leq_{\mathscr{T}} \qquad \text{and} \qquad \pi^*(\leq) = \mathscr{A}_\leq$$

Here π is a reflection, i.e., every pre-order is π-open, and π is an isomorphism iff X is finite. The π-closed topologies are precisely the Alexandroff topologies; the corresponding lattice is isomorphic to the lattice of all pre-orders on X ordered by reverse inclusion.

Among the topologies $\mathscr{T}$ with specialization order $\leq$ those with the property that every $\leq$-directed set with a least upper bound in $U \in \mathscr{T}$ is eventually in U are called **order-consistent**; the finest such topology is the **Scott topology** $\mathscr{S}_\leq$. It consists of all those $U \subseteq X$ that satisfy $x \in U \iff U \cap D \neq \emptyset$ for every $\leq$-directed $D \subseteq X$ with a least upper bound x (cf. [25] and [19]). (Note that "$\Leftarrow$" above implies $U = \uparrow U$.) In fact, all topologies between $\mathscr{U}_\leq$ and $\mathscr{S}_\leq$ are order-consistent. Hence we may restrict π_* to the subposet $\mathscr{P}'$ of order-consistent topologies on X to obtain a new Galois connection π' with $\pi'^*(\leq) = \mathscr{S}_\leq$. Now the π'-closed elements of $\mathscr{P}'$ are precisely the Scott topologies (cf. [41]). $\square$

Example 10: Let $\mathscr{P}$ be the set of all equivalence relations on a set X, ordered by inclusion, and let $\mathscr{Q}$ be the class of all functions with domain X, ordered by $(X \xrightarrow{f} Y) \sqsubseteq (X \xrightarrow{h} Z)$ iff there exists some function $Y \xrightarrow{g} Z$ that satisfies $g \circ f = h$. Define π_* and π^* by setting

$$\pi_*(R) = (X \xrightarrow{p_R} X/R) \qquad \text{(the canonical surjection)}$$
$$\pi^*(f) = \boldsymbol{ker}(f) = \{\langle x, y\rangle \in X \times X \mid f(x) = f(y)\}$$

This is not a Galois connection as defined in Definition 1 for two reasons:

- $\mathscr{Q}$ is a proper class rather than a set;

- the order on $\mathscr{Q}$ is only a pre-order, rather than a partial order, i.e., it lacks antisymmetry.

However, neither of these problems creates a serious roadblock. Galois connections can be defined for both proper classes and pre-orders without any changes; however some of the consequences and notions in Propositions 3 – 6 require slight modifications. For example, rather than $\pi_* = \pi_* \circ \pi^* \circ \pi_*$ in Proposition 3(2) we have $\pi_* \cong \pi_* \circ \pi^* \circ \pi_*$ in the sense that $\pi_* \sqsubseteq \pi_* \circ \pi^* \circ \pi_* \sqsubseteq \pi_*$. Also an element $p \in P$ is π-closed iff $p \leq \pi^*\pi_*(p) \leq p$, and an element $q \in Q$ is π-open iff $q \sqsubseteq \pi_*\pi^*(q) \sqsubseteq q$. Etc.

In this example the π-open functions with domain X are precisely the surjective ones, and all equivalence relations on X are π-closed (so π is a coreflection). This makes precise the usual interplay between equivalence relations and surjective functions. □

We conclude our list of examples with two related constructions that establish interesting links between order theory, topology, algebra and computer science.

EXAMPLE 11: (cf. [58], [17], [20]) Let $\mathscr{Z}$ denote an **R-invariant extension**, that is, a function assigning to each poset $\mathscr{P}$ a certain collection $\mathscr{Z}(\mathscr{P})$ of lower sets including all principal ideals such that every residuated map $\mathscr{P} \xrightarrow{f} \mathscr{Q}$ is **$\mathscr{Z}$-continuous**, i.e., $V \in \mathscr{Z}(\mathscr{Q})$ implies $f^{\leftarrow}(V) \in \mathscr{Z}(\mathscr{P})$. Equivalently, this means that every residual map $\mathscr{Q} \xrightarrow{g} \mathscr{P}$ is **$\mathscr{Z}$-quasi-closed**, i.e., $U \in \mathscr{Z}(\mathscr{P})$ implies $\downarrow g_{\rightarrow}(U) \in \mathscr{Z}(\mathscr{Q})$. Possible choices for $\mathscr{Z}$ are: principal ideals, arbitrary lower sets, finitely generated lower sets, directed lower sets, lower cuts (see Example 25), ideals, Scott-closed sets (see Example 9), etc.

An element p of a **$\mathscr{Z}$-complete** poset $\mathscr{P}$ (where each $Z \in \mathscr{Z}(\mathscr{P})$ has a join = supremum) is called **$\mathscr{Z}$-compact** or **$\mathscr{Z}$-prime** if it belongs to every $Z \in \mathscr{Z}(\mathscr{P})$ whose join dominates p, and $\mathscr{P}$ is called **$\mathscr{Z}$-compactly generated** if every element of $\mathscr{P}$ is a join of $\mathscr{Z}$-compact elements. By a **$\mathscr{Z}$-lattice** we mean a $\mathscr{Z}$-compactly generated complete lattice. Of fundamental importance for various topological representation theories are the following two facts:

(A) *The $\mathscr{Z}$-lattices are, up to isomorphism, precisely the closure systems that are closed under $\mathscr{Z}$-unions.*

This generalizes the known facts that the compactly generated complete lattices are represented by algebraic closure systems, the $\vee$-primely generated complete lattices are represented by lattices of closed sets in topological spaces, etc.

(B) *A coadjoint map between $\mathscr{Z}$-compactly generated posets preserves $\mathscr{Z}$-compactness iff its adjoint preserves $\mathscr{Z}$-joins.*

By definition, a map $\mathscr{P} \xrightarrow{f} \mathscr{Q}$ preserves $\mathscr{Z}$-compactness iff it maps $\mathscr{Z}$-compact elements of $\mathscr{P}$ to $\mathscr{Z}$-compact elements of $\mathscr{Q}$, and a map $\mathscr{Q} \xrightarrow{g} \mathscr{P}$ preserves $\mathscr{Z}$-joins iff $g(\bigvee Z) = \bigvee g_{\rightarrow} Z$ for all $Z \in \mathscr{Z}(\mathscr{Q})$.

By combining (A) with the hyperspace construction of Example 4, one obtains an equivalence between the category of **$\mathscr{Z}$-sober** closure spaces (i.e., T_0 closure spaces where the point closures are precisely the $\mathscr{Z}$-compact *closed* sets) with continuous maps, and the category of $\mathscr{Z}$-lattices with maps preserving joins and $\mathscr{Z}$-compactness. By (B) these categories are *dual* to the category of $\mathscr{Z}$-lattices and maps preserving meets and $\mathscr{Z}$-joins.

In case $\mathscr{Z}$ selects the finitely generated lower sets, we arrive at the known duality between sober spaces and (dual) spatial frames (see, e.g., [25]). On the other hand, for the special selection of all directed lower sets ("ideals" in the sense of [25]), these considerations reduce to known dualities between algebraic lattices and (compact) semilattices, etc. For reasons of limited space, we refer the reader to the literature, especially to Hofmann and Stralka ([32]); throughout that study on compact topological semilattices, a consistent theme is the continued application of Galois connections (see also Example 12).

Now suppose that *two* R-invariant extensions $\mathscr{Z}$ and $\mathscr{Z}'$ are given.

Let $\mathscr{Z}\boldsymbol{S}\mathscr{Z}'$ be the category whose objects are the $\mathscr{Z}$-sober closure spaces with a base of $\mathscr{Z}'$-compact *open* sets, and whose morphisms are maps such that inverse images of $\mathscr{Z}'$-compact open sets are again $\mathscr{Z}'$-compact and open (in particular, such maps are continuous).

Let $\mathscr{Z}\boldsymbol{C}\mathscr{Z}'$ be the category whose objects are those $\mathscr{Z}$-lattices whose duals are $\mathscr{Z}'$-lattices, and whose morphisms are maps that preserve arbitrary joins, $\mathscr{Z}$-compactness, and $\mathscr{Z}'$-meets.

Because of (B) it is straightforward to show that order-dualizing the objects and passing to adjoint morphisms yields a *dual isomorphism between the categories* $\mathscr{Z}\boldsymbol{C}\mathscr{Z}'$ *and* $\mathscr{Z}'\boldsymbol{C}\mathscr{Z}$, and this is the root for many other dualities. Indeed, the aforementioned hyperspace construction leads to an equivalence between $\mathscr{Z}\boldsymbol{S}\mathscr{Z}'$ and $\mathscr{Z}\boldsymbol{C}\mathscr{Z}'$ as well as to one between $\mathscr{Z}'\boldsymbol{S}\mathscr{Z}$ and $\mathscr{Z}'\boldsymbol{C}\mathscr{Z}$. By composing these with the duality above, one arrives at a nice "symmetric" duality between the categories $\mathscr{Z}\boldsymbol{S}\mathscr{Z}'$ and $\mathscr{Z}'\boldsymbol{S}\mathscr{Z}$. Now, special choices of $\mathscr{Z}$ and $\mathscr{Z}'$ and suitable restrictions provide a multitude of "classical" dualities, encompassing the dualities between

- sober spaces and spatial frames [25], [35]
- sober spaces with compact-open base and distributive semilattices [28]
- Boolean spaces (Stone spaces) and Boolean lattices [54], [35]
- semilattices and algebraic lattices [24], [32]
- Alexandroff spaces (posets) and completely distributive algebraic lattices [16]

and many others, including Lawson's duality for continuous, respectively, algebraic posets (see [38] and Example 12 below). For details consult [20]. □

EXAMPLE 12: (cf. [43], [5], [55]) A vital theme of modern order theory is **continuous posets**, and more generally, $\mathscr{Z}$**-continuous posets** (where $\mathscr{Z}$ denotes, as in the previous example, an R-invariant extension). Such posets may be characterized most conveniently in terms of Galois connections: a poset $\mathscr{P} = \langle P, \leq \rangle$ is $\mathscr{Z}$**-complete** iff the principal ideal embedding of $\mathscr{P}$ into $\mathscr{Z}(\mathscr{P})$ given by $x \mapsto \downarrow x$ has a coadjoint (namely the join map $\bigvee$ from $\mathscr{Z}(\mathscr{P})$ to $\mathscr{P}$). A $\mathscr{Z}$-complete poset $\mathscr{P}$ is $\mathscr{Z}$**-continuous** iff its join map $\bigvee$ is also adjoint (to the $\mathscr{Z}$**-below map** $x \mapsto \downarrow_{\mathscr{Z}} x = \bigcap\{ Z \in \mathscr{Z}(\mathscr{P}) \mid x \leq \bigvee \mathscr{Z} \}$). This approach makes many considerations on $\mathscr{Z}$-continuous posets short, elegant, and transparent.

A "good" class of morphisms is certainly formed by coadjoint maps that preserve the $\mathscr{Z}$**-below relation** (i.e., that satisfy $f_{\rightarrow}(\downarrow_{\mathscr{Z}} x) \subseteq \downarrow_{\mathscr{Z}} f(x)$); via Galois connections, the corresponding category is dually isomorphic to the category of $\mathscr{Z}$-continuous posets and adjoint maps preserving $\mathscr{Z}$-joins. Full subcategories are formed by the $\mathscr{Z}$**-algebraic posets**, i.e.,

$\mathscr{Z}$-continuous and $\mathscr{Z}$-compactly generated posets (cf. Example 11). Since an element p is $\mathscr{Z}$-compact iff $p \in \downarrow_{\mathscr{Z}} p$, preservation of the $\mathscr{Z}$-below relation here is tantamount to preservation of $\mathscr{Z}$-compactness.

If $\mathscr{Z}$ selects all lower sets, i.e., $\mathscr{Z}\langle P, \leq\rangle$ is the dual Alexandroff topology $\mathscr{A}_{\geq}$ (cf. Example 9), then the $\mathscr{Z}$-continuous posets are just the **completely distributive** lattices (cf. [48]). Hence a complete lattice $\mathscr{P}$ is completely distributive iff the join map $\bigvee$ from $\mathscr{A}_{\geq}$ to $\mathscr{P}$ is a complete homomorphism.

An easy Galois argument (explained in greater detail in Example 31) leads to the following result on so-called **$\mathscr{Z}$-join ideals**, i.e., fixed points of the operator

$$\Delta_{\mathscr{Z}} : Y \mapsto \downarrow\{ \bigvee Z \mid Z \in \mathscr{Z}(\mathscr{P}) \ , Z \subseteq \downarrow Y \}$$

- *For any $\mathscr{Z}$-continuous poset with idempotent $\mathscr{Z}$-below relation, the closure system of $\mathscr{Z}$-join ideals is completely distributive, and $\Delta_{\mathscr{Z}}$ is the corresponding closure operator.*

For many R-invariant extensions occurring in practice (e.g., for each of the selections mentioned at the beginning of Example 11, and more generally, for each $\mathscr{Z}$ such that $\bigcup \mathscr{Y} \in \mathscr{Z}(\mathscr{P})$ whenever $\mathscr{Y} \in \mathscr{Z}\langle \mathscr{Z}(\mathscr{P}), \subseteq\rangle$), *every* $\mathscr{Z}$-continuous poset has an idempotent $\mathscr{Z}$-below relation (see [5]). For example, if $\mathscr{Z}$ selects the directed lower sets, then "$\mathscr{Z}$-continuous" means "continuous", "$\mathscr{Z}$-algebraic" means "algebraic", and the $\mathscr{Z}$-join ideals are just the Scott-closed sets. By the previous remarks, they form a completely distributive lattice for any continuous poset (cf. [38] and [25]). $\square$

RESIDUATED SEMIGROUPS

In this section we briefly discuss a general concept that accounts for a large class of interesting Galois connections.

DEFINITION 3: A partially ordered semigroup $\langle P, \leq, \cdot\rangle$, i.e., a poset $\mathscr{P} = \langle P, \leq\rangle$ equipped with an order-preserving associative multiplication $\mathscr{P} \times \mathscr{P} \stackrel{\cdot}{\longrightarrow} \mathscr{P}$, is called **residuated** if all left translations $\mathscr{P} \xrightarrow{r\cdot -} \mathscr{P}$ and all right translations $\mathscr{P} \xrightarrow{-\cdot s} \mathscr{P}$ are residuated, i.e., have adjoints $\mathscr{P} \xrightarrow{r\backslash -} \mathscr{P}$ and $\mathscr{P} \xrightarrow{-/s} \mathscr{P}$, respectively; this means

$$s \leq r\backslash t \iff r \cdot s \leq t \iff r \leq t/s$$

Notice that, in addition to the Galois connections $\langle r \cdot -, r\backslash -\rangle$ and $\langle - \cdot s, -/s\rangle$ from $\mathscr{P}$ to $\mathscr{P}$ given by the left and right translations above, for every $t \in P$ the pair $\langle -\backslash t, t/-\rangle$ is a Galois connection from $\mathscr{P}$ to $\mathscr{P}^{\mathrm{op}}$.

By Proposition 4(2) a complete partially ordered semigroup $\langle \mathscr{P}, \cdot\rangle$ is residuated iff the multiplication distributes over arbitrary suprema, i.e., for all $r \in P$ and $S \subseteq P$

$$r \cdot \bigvee S = \bigvee\{ r \cdot s \mid s \in S \} \qquad \text{and} \qquad \bigvee S \cdot r = \bigvee\{ s \cdot r \mid s \in S \}$$

Nowadays, complete residuated semigroups are referred to as **quantales**. For a comprehensive treatise on this modern branch of order theory, see the text by Rosenthal [50]. A classical source on residuation theory is the monograph by Blyth and Janowitz [9].

EXAMPLE 13: (cf. [18]) Let $\langle A, \cdot \rangle$ be a semigroup, and let $\mathscr{X}$ be a closure system on A that makes the left and right translations continuous. The complete lattice $\langle \mathscr{X}, \subseteq \rangle$ becomes a quantale when equipped with the multiplication $R \odot S = \{ r \cdot s \mid r \in R,\ s \in S \}^-$. The quantale $\langle \mathscr{X}, \subseteq, \odot \rangle$ may be regarded as a "residuated completion" of the "generalized semitopological semigroup" $\langle A, \mathscr{X}, \cdot \rangle$. By Example 4, every continuous semigroup homomorphism lifts to a residuated semigroup homomorphism between the associated completions. These facts provide a broad spectrum of useful constructions for order-theoretical and topological completions. Among these, we only mention two specific examples: the MacNeille completion of a residuated semigroup, and the Vietoris hyperspace of a continuous semilattice. For more details see [21] and [18]. $\square$

EXAMPLE 14: In ring theory, the closure systems of additive subgroups, of left ideals, of right ideals, and of two-sided ideals of a given ring A each play a major role. Endowed with the multiplication $RS = \{ \sum_{i=0}^{n} r_i s_i \mid r_i \in R,\ s_i \in S,\ n \in \omega \}$, each of these closure systems becomes a quantale. However, only in the case of two-sided ideals are there "natural" expressions for both the left and right residual maps given by

$$R \backslash T = \{ a \in A \mid \forall_{r \in R}\ ra \in T \} \qquad \text{and} \qquad T/S = \{ a \in A \mid \forall_{s \in S}\ as \in T \}$$

More topologically flavored variants of these examples are the quantales of all left, right, or two-sided closed ideals, respectively, of a $C^\star$-algebra. $\square$

EXAMPLE 15: By a partial semigroup $\mathscr{A}$, we mean a set A equipped with a partial binary operation $\cdot$ such that whenever $r \cdot s$ and $s \cdot t$ are defined, then so are $(r \cdot s) \cdot t$ and $s \cdot (r \cdot t)$, and these products are equal. Any such $\mathscr{A}$ induces a quantale $\langle \boldsymbol{P}(A), \otimes \rangle$ via

$$\begin{aligned} R \otimes S &= \{ r \cdot s \mid r \in R, s \in S, \text{ and } r \cdot s \text{ is defined} \} \\ R \backslash T &= \{ a \in A \mid \forall_{r \in R} \text{ if } r \cdot a \text{ is defined then } r \cdot a \in T \} \\ T/S &= \{ a \in A \mid \forall_{s \in S} \text{ if } a \cdot s \text{ is defined then } a \cdot s \in T \} \end{aligned}$$

Important subquantales of $\langle \boldsymbol{P}(A), \otimes \rangle$ are the Alexandroff topologies of all left, right or two-sided ideals of A, respectively. $\square$

EXAMPLE 16: Let Σ be a set (often called an **alphabet**), and let $\Sigma^\star$ be the free monoid of Σ-**words** with concatenation as the operation. Subsets of $\Sigma^\star$ are called **languages** over Σ. By the previous remarks the languages form a quantale, and consequently every language L gives rise to three Galois connections $\langle L \otimes -, L \backslash - \rangle$, $\langle - \otimes L, -/L \rangle$, and $\langle - \backslash L, L/- \rangle$. $\square$

EXAMPLE 17: Define a partial semigroup operation $\cdot$ on $A \times A$ by setting $\langle a, b \rangle \cdot \langle d, c \rangle = \langle a, c \rangle$ iff $b = d$. Then for relations $R, S \subseteq A \times A$ the product $R \otimes S$ is the usual composite relation $\{ \langle a, c \rangle \mid \exists_b \text{ with } \langle a, b \rangle \in R \text{ and } \langle b, c \rangle \in S \}$. So $\langle \boldsymbol{P}(A \times A), \otimes \rangle$ is a quantale. The next two sections deal explicitly with Galois connections induced by relations.

EXAMPLE 18: (cf. [26], [8], [23], [49], [28], [37]) Let p be a fixed element of a meet-semilattice $\mathscr{P} = \langle P, \leq \rangle$. We say $\mathscr{P}$ is **p-pseudocomplemented** if every element $r \in P$ has a **p-pseudocomplement** r^p (also denoted by $r * p$, $r \to p$, $r \backslash p$, p/r, $p : r$, etc.) that satisfies $r \wedge s \leq p$ iff $s \leq r^p$. Since $\wedge$ is symmetric, $\pi_*(r) = \pi^*(r) = r^p$ defines a Galois connection $\mathscr{P} \xrightarrow{\pi} \mathscr{P}^{\mathrm{op}}$. Hence the **$p$-skeleton**

$$P^\pi = P^p = \{ r^p \mid r \in P \} = \{ r \in P \mid r = r^{pp} \}$$

is a closure system of $\mathscr{P}$; in particular, $\mathscr{P}^p = \langle P^p, \leq \rangle$ is a meet-subsemilattice of $\mathscr{P}$, and if $\mathscr{P}$ is complete, so is $\mathscr{P}^p$. The meet-semilattice $\mathscr{P}$ is called **Brouwerian** or **relatively pseudocomplemented** iff it is p-pseudocomplemented for each $p \in P$. This is equivalent to saying that each of the unary meet operations $\mathscr{P} \xrightarrow{r \wedge -} \mathscr{P}$ is residuated (with adjoint $\mathscr{P} \xrightarrow{r * -} \mathscr{P}$). Hence the Brouwerian semilattices are those residuated semigroups whose multiplication is the binary meet. Brouwerian lattices with least element are also known as **Heyting algebras**, and complete Heyting algebras are known as **frames** or **locales**. Since residuated maps preserve joins, every Brouwerian lattice is distributive, and the locales are precisely those complete lattices which satisfy the infinite distributive law

$$r \wedge \bigvee S = \bigvee \{ r \wedge s \mid s \in S \}$$

Lattices of this type play a considerable role in disciplines as diverse as congruence theory, (intuitionistic) logic, and topology. In fact, residuation yields a duality between the category of frames (with maps preserving arbitrary joins and finite meets) and the category of locales (with maps adjoint to frame morphisms), which in turn contains (an isomorphic copy of) the category of topological spaces (cf. Example 11). As was pointed out by Isbell [34], Banaschewski and Pultr [4], Johnstone [36] and others, this aspect is of fundamental importance for a general theory of "spaces without points".

Moreover, Galois connections of the present type produce Boolean lattices in abundance:

- *Any p-pseudocomplemented meet-semilattice $\mathscr{P}$ has a p-skeleton $\mathscr{P}^p$ that is a Boolean lattice with least element p and greatest element p^p.*

To verify this claim, three simple observations suffice: for $r, s \in P^p$

(1) $(r^p \wedge s^p)^p$ is the join in $\mathscr{P}^p$;

(2) r^p is the complement of r in $\mathscr{P}^p$,

(3) $(r \wedge s^p)^p$ is the relative pseudocomplement in $\mathscr{P}^p$ (in particular, $\mathscr{P}^p$ is distributive).

In case p is the least element $\perp$, the above notions reduce to **pseudocomplement**, **pseudocomplemented (semi-) lattice**, and **skeleton**, respectively.

Since every topology $\mathscr{T}$ is a locale with pseudocomplements $U^\perp = U^{-\prime}$, one immediately concludes that the regular open sets form a Boolean lattice, namely the skeleton of $\mathscr{T}$ (cf. Example 5). Compare this concise argument with, e.g., the treatment in [29]. □

POLARITIES

All Galois connections between power sets are induced by relations between the underlying sets in particularly simple ways. For sets A and B we first study the contravariant case of Galois connections from $\boldsymbol{P}(A)$ to $\boldsymbol{P}(B)^{\mathrm{op}}$, so-called **polarities** between $P(A)$ and $P(B)$. (Recall, however, that many authors use the term "Galois connection" exclusively in the contravariant sense.) Since the π-open subsets of such a Galois connection π form a *closure* system *on* B, the term "π^{op}-closed" is preferable in this context.

PROPOSITION 7: (cf. [8], [44]) (1) *Any relation $R \subseteq A \times B$ induces a Galois connection $\boldsymbol{P}(A) \xrightarrow{R_+{}^+} \boldsymbol{P}(B)^{\mathrm{op}}$. The components of $R_+{}^+ = \langle R_+, R^+\rangle$ are defined by*

$$R_+(U) := \{\, b \in B \mid \forall_{a\in U}\ \langle a,b\rangle \in R \,\} \qquad \text{for } U \subseteq A$$
$$R^+(V) := \{\, a \in A \mid \forall_{b\in V}\ \langle a,b\rangle \in R \,\} \qquad \text{for } V \subseteq B$$

(2) *If for a polarity $\boldsymbol{P}(A) \xrightarrow{\pi} \boldsymbol{P}(B)^{\mathrm{op}}$ the relation $|\pi| \subseteq A \times B$ is defined by $\langle a,b\rangle \in |\pi|$ iff $b \in \pi_*(\{a\})$ (or, equivalently, $a \in \pi^*(\{b\})$), then $\pi = |\pi|_+{}^+$.*

(3) *Every relation $R \subseteq A \times B$ satisfies $|R_+{}^+| = R$. Hence every polarity π between $P(A)$ and $P(B)$ comes from a unique relation, namely $R = |\pi|$, and vice versa.*

(4) *$R^+{}_+ := (R_+{}^+)^{\mathrm{op}} = \langle R^+, R_+\rangle$ is the polarity between $P(B)$ and $P(A)$ induced by the* **opposite relation** $R^{\mathrm{op}} = \{\, \langle b,a\rangle \in B \times A \mid \langle a,b\rangle \in R \,\}$. □

The previous considerations admit a natural generalization. Let P be a closure system on A, and let Q be a closure system on B. By a **polarity** between P and Q we mean a Galois connection between the complete lattices $\langle P, \subseteq\rangle$ and $\langle Q, \supseteq\rangle$. Since every complete lattice is isomorphic to the closure system of all principal ideals, and is dually isomorphic to the closure system of all principal filters, essentially *all* Galois connections between complete lattices may be regarded as certain polarities. The polarities between P and Q form a complete lattice with respect to the pointwise ordering, called the **tensor product** of P and Q (see, e.g., [52]). By Propositions 6 and 7, every polarity between P and Q is induced by a unique relation $R \subseteq A \times B$. Hence the interior system of all relations $R \subseteq A \times B$ with $R_{+\to}(P(A)) \subseteq Q$ and $R^+{}_\to(P(B)) \subseteq \mathscr{P}$ is dually isomorphic to the tensor product of P and Q.

EXAMPLE 19: The classical Galois connection of Example 1 is induced by the relation $R = \{\, \langle a,g\rangle \in E \times G \mid g(a) = a \,\}$. The $R_+{}^+$-closed sub*sets* of E are precisely the π-closed intermediate *fields* of $E : F$, and the $R^+{}_+$-closed sub*sets* of G are precisely the π^{op}-closed sub*groups* of G, cf. Proposition 6. □

EXAMPLE 20: (cf. [15]) For $A = P(X)$ and $B = X \times X$, the Alexandroff reflection of Example 9 is induced by the relation $R = \{\, \langle U, \langle x,y\rangle\rangle \in A \times B \mid x \in U \text{ or } y \in U \,\}$. For $\mathscr{U} \subseteq P(X)$ the **specialization order** $R_+(\mathscr{U}) = \leq_{\mathscr{U}}$ is defined in analogy to formula (4). The adjoint R^+ maps a relation S on X to the Alexandroff topology $\mathscr{A}_S$ of all right S-closed sets, i.e., $U \in R_+(S) = \mathscr{A}_S$ iff $x \in U$ and $\langle x,y\rangle \in S$ together imply $y \in U$. Thus $R^+R_+(\mathscr{U})$ is the Alexandroff topology generated by $\mathscr{U}$, while $R_+R^+(S)$ is the reflexive transitive closure of S.

Hence the ${R_+}^+$-closed subsets of A are precisely the Alexandroff topologies on X, and the ${R^+}_+$-closed subsets of B are precisely the pre-orders on X. $\square$

EXAMPLE 21: (cf. [53]) For the set $\boldsymbol{F}(X)$ of all filters on X consider the relation $R \subseteq P(X) \times (X \times \boldsymbol{F}(X))$ given by $\langle U, \langle x, F\rangle\rangle \in R \iff x \notin U$ or $U \in F$. The ${R_+}^+$-closed subsets of A are precisely the topologies on X. The ${R^+}_+$-closed subsets of $X \times \boldsymbol{F}(X)$ are precisely the topological **convergence relations**, i.e., the relations $C \subseteq X \times \boldsymbol{F}(X)$ for which there exists a topology $\mathscr{T}$ on X such that a filter F $\mathscr{T}$-converges to a point x iff $\langle x, F\rangle \in C$. This polarity makes precise the interplay between topologies and filter convergence.

The relation R of Example 20 can be viewed as a restriction of the current relation R if points are identified with principal ultrafilters. By restricting from arbitrary filters to principal filters (which can be identified with subsets of X), one obtains a polarity whose closed subsets of $X \times P(X)$ are precisely the topological **adherence relations**. $\square$

EXAMPLE 22: (cf. [46], [2]) Let R be the relation on the class of all topological spaces defined by $\langle X, Y\rangle \in R$ iff all continuous functions from X to Y are constant. Although this is a relation on a proper class we still obtain a polarity on the collection of all subclasses of the class of topological spaces. Given a class $\mathscr{E}$ of topological spaces, the spaces in $R^+(\mathscr{E})$ are called **$\mathscr{E}$-connected**, and the spaces in $R_+R^+(\mathscr{E})$ are called **$\mathscr{E}$-disconnected**. E.g., if 2 denotes a two-element discrete space, then "$\{2\}$-connected" means "connected" in the usual sense, and "$\{2\}$-disconnected" means "totally disconnected". $\square$

EXAMPLE 23: Let A be the class of all continuous functions, and let B be the class of all topological spaces. Define relations R and S between continuous functions and topological spaces as follows: $\langle X \xrightarrow{f} Y, Z\rangle \in R$ (resp. $\langle X \xrightarrow{f} Y, Z\rangle \in S$) iff for every continuous function $X \xrightarrow{h} Z$ there is at least one (resp. at most one) continuous function $Y \xrightarrow{g} Z$ with $h = g \circ f$. Then R_+ maps a class $\mathscr{M}$ of continuous functions to the class of so-called **$\mathscr{M}$-injective spaces**. The class $S_+(\mathscr{M})$ consists of the **$\mathscr{M}$-separated spaces** in the sense of Pumplün and Röhrl [47]. For example, if $\mathscr{M}$ is the class of all dense embeddings, then $S_+(\mathscr{M})$ is precisely the class of Hausdorff spaces. On the other hand, $S^+(B)$ is the class of all surjective continuous functions. A dual construction for R yields **$\mathscr{M}$-projective spaces**. All these relations have analogs that yield useful polarities in general categories. A generalization of the **orthogonality relation** $R \cap S$ is used extensively in [11]. $\square$

EXAMPLE 24: (cf. Example 15) If $\mathscr{A} = \langle A, \cdot\rangle$ is a partial semigroup, then for $T \subseteq A$ the polarity $\boldsymbol{P}(A) \xrightarrow{\langle -\backslash T, T/-\rangle} \boldsymbol{P}(A)^{\mathrm{op}}$ is generated by the relation $\{\langle a, b\rangle \in A \times A \mid a \cdot b \text{ is defined and } a \cdot b \in T\}$. $\square$

EXAMPLE 25: (cf. [56]) A "restructured" view of polarities with applications in extra-mathematical disciplines is pursued in the modern theory of *Formal Concept Analysis*. Here one studies a set A of "objects", a set B of "attributes", and a relation $R \subseteq A \times B$. The triple $\langle A, B, R\rangle$ is interpreted as a **context**. Pairs of the form $\langle U, V\rangle \in P(A) \times P(B)$ with $R_+(U) = V$ and $R^+(V) = U$ are called **concepts**; U is the **extent** and V is the **intent** of the concept $\langle U, V\rangle$. The concepts form a complete lattice, isomorphic to the closure system of all extents (= ${R_+}^+$-closed subsets of A ordered by inclusion), and dually isomorphic to the closure system of all intents (= ${R^+}_+$-closed subsets of B ordered by inclusion). A thorough investigation of the concept lattice reveals implications and dependencies between the various concepts involved.

If R is a pre-order $\leq$ on a set P, then $R_+(U)$ is the set of all **upper** (!) **bounds** and $R^+(U)$ is the set of all **lower** (!) **bounds** for $U \subseteq P$. The concept lattice of the context $\langle P, P, \leq \rangle$ is known as the **Dedekind-MacNeille completion** or the **completion by cuts** for the pre-ordered set $\mathscr{P} = \langle P, \leq \rangle$ (cf. [40] and [8]). It is well-known that a complete lattice $\mathscr{L}$ is a **normal completion** of $\mathscr{P}$ (that is, $\mathscr{P}$ admits a join- and meet-dense embedding into $\mathscr{L}$) iff $\mathscr{L}$ is isomorphic to the completion by cuts of $\mathscr{P}$. Two further examples of such normal completions for $\mathscr{P}$ are the systems of all ${R_+}^+$-closed sets (**lower cuts**), ordered by inclusion, and the system of all ${R^+}_+$-closed sets (**upper cuts**), ordered by reverse inclusion. □

EXAMPLE 26: As before, let $\mathscr{P} = \langle P, \leq \rangle$ be a pre-ordered set, but let R be the complementary dual relation $\not\geq$. It is easy to see that $R_+(U)$ is the complement in P of the upper set $\uparrow U$ generated by U, while $R^+(V)$ is the complement of the **lower set** $\downarrow V$ (see Example 9). Thus the isomorphism between the Alexandroff topology of all upper (= ${R_+}^+$-open) sets and that of all lower (= ${R_+}^+$-closed) sets via complementation is induced by the relation R. The associated closure operations preserve arbitrary unions.

If R is the inequality relation $\neq$ on a set A, then ${R_+}^+$ is the isomorphism from $\boldsymbol{P}(A)$ to $\boldsymbol{P}(A)^{\mathrm{op}}$ given by complementation. We denote its inverse by $\boldsymbol{P}(A)^{\mathrm{op}} \xrightarrow{\kappa_A} \boldsymbol{P}(A)$. □

EXAMPLE 27: Consider a meet-semilattice $\mathscr{P} = \langle P, \leq \rangle$ with least element $\perp$, and the relation $R = \{ \langle a, b \rangle \in P \times P \mid a \wedge b = \perp \}$. For any $U \subseteq P$ the set $U^\perp = R_+(U) = R^+(U)$ is a lower set. If $\mathscr{P}$ is pseudocomplemented (cf. Example 18), then $U^\perp$ is even a lower cut, namely the set of lower bounds for all pseudocomplements of members of U. Any two lower sets U and V satisfy $U \cap V = \{\perp\}$ iff $U \subseteq V^\perp$, which explicitly describes the pseudocomplementation in the frame of lower sets. Hence the closure system $\boldsymbol{S}(\mathscr{P})$ of all ${R_+}^+$-closed (= ${R_+}^+$-open) sets is the skeleton of this frame (see again Example 18), and as such it is Boolean. It is now easy to see that $\boldsymbol{S}(\mathscr{P})$ is actually a normal completion of the skeleton $\mathscr{P}^\perp$. In particular this shows that Boolean lattices have Boolean completions by cuts. □

EXAMPLE 28: Again, let $\mathscr{P} = \langle P, \leq \rangle$ be a meet-semilattice with least element $\perp$, but this time consider $R = \{ \langle a, b \rangle \in P \times P \mid a \wedge b \neq \perp \}$. Here the ${R_+}^+$-closed (= ${R_+}^+$-open) sets are upper sets. If $\mathscr{P}$ is even a Boolean lattice, then *every* upper set is ${R_+}^+$-closed; the ${R_+}^+$-closure of $U \subseteq P$ is just the upper set $\uparrow U$. In this specific situation by Proposition 3(5) the Alexandroff topology $\mathscr{A}_\leq$ is actually self-dual, as well as dual to $\mathscr{A}_\geq$.

In particular, this applies to any power set lattice $\boldsymbol{P}(X)$. Here $R_+ = R^+$ is the **section operator** considered by Choquet [12] and others (see, e.g., [51] and [30]) in connection with convergence-related questions: $R_+(\mathscr{U}) = \mathscr{U}^\sharp = \{ V \subseteq X \mid \forall_{U \in \mathscr{U}}\ U \cap V \neq \emptyset \}$. Moreover, $\mathscr{U}^{\sharp\sharp}$ is the **stack** generated by $\mathscr{U}$, i.e., the collection of all sets V with $U \subseteq V \subseteq X$ for some $U \in \mathscr{U}$. The well-known isomorphism between the lattice of filters on X and the lattice of **grills** on X is obtained by a suitable restriction and corestriction of R_+. The ultrafilters are precisely the fixed points of this dual isomorphism. This puts the interplay between limits and cluster points into a nice symmetric framework: a point x is a cluster point of a stack $\mathscr{U}$ (in some topological space or convergence space) iff x is a limit of the stack $\mathscr{U}^\sharp$.

The section operator has also been used to give a symmetric formulation of the notion of complete distributivity: a complete lattice $\langle X, \leq\rangle$ is completely distributive iff for every collection $\mathscr{U}$ of subsets of X the following identity holds:

$$\bigwedge\Big\{ \bigvee U \mid U \in \mathscr{U} \Big\} = \bigvee\Big\{ \bigwedge V \mid V \in \mathscr{U}^\sharp \Big\}$$

Since replacing $\mathscr{U}$ with the stack $\mathscr{U}^{\sharp\sharp}$ does not change either side of the equation, we immediately see that complete distributivity is a self-dual property (see also Example 12). □

AXIALITIES

We now consider the covariant case of Galois connections between power sets both ordered by inclusion; we call such Galois connections **axialities**. The strong similarity between the following result and Proposition 7 despite the different orders on the second power set is quite surprising.

PROPOSITION 8: (cf. [27]) (1) *Any relation* $R \subseteq A \times B$ *induces a Galois connection* $\boldsymbol{P}(A) \xrightarrow{R_\exists{}^\forall} \boldsymbol{P}(B)$. *The components of* $R_\exists{}^\forall = \langle R_\exists, R^\forall\rangle$ *are defined by*

$$R_\exists(U) := \{\, b \in B \mid \exists_{a\in A}\ \langle a,b\rangle \in R \text{ and } a \in U \,\} \qquad \text{for } U \subseteq A$$
$$R^\forall(V) := \{\, a \in A \mid \forall_{b\in B}\ \langle a,b\rangle \in R \text{ implies } b \in V \,\} \qquad \text{for } V \subseteq B$$

(2) *If for an axiality* $\boldsymbol{P}(A) \xrightarrow{\pi} \boldsymbol{P}(B)$ *the relation* $|\pi| \subseteq A \times B$ *is defined by* $\langle a,b\rangle \in |\pi|$ *iff* $b \in \pi_*(\{a\})$, *then* $\pi = |\pi|_\exists{}^\forall$.

(3) *Every relation* $R \subseteq A \times B$ *satisfies* $R = |R_\exists{}^\forall|$. *Hence every axiality* π *from* $\boldsymbol{P}(A)$ *to* $\boldsymbol{P}(B)$ *comes from a unique relation* $R \subseteq A \times B$, *namely* $R = |\pi|$, *and vice versa.*

(4) $R^\exists{}_\forall = \langle R^\exists, R_\forall\rangle := (R^{\mathrm{op}})_\exists{}^\forall$ *is an axiality from* $\boldsymbol{P}(B)$ *to* $\boldsymbol{P}(A)$. □

As in the contravariant case, any Galois connection between a closure system on A and an interior system on B is induced by a unique axiality, hence by a unique relation $R \subseteq A \times B$ (cf. Propositions 6 and 8).

The following theorem provides a common generalization of several results encountered so far. It is based on the phenomenon, first recognized by Lawvere [39], that quantification is adjoint to substitution. Specifically, it shows that each relation between two sets not only induces a whole family of polarities, but also two families of axialities, from which the polarities arise as a derived concept. By substituting singletons for A, C, and B in parts (1), (2) and (3) of the theorem, respectively, one obtains (after suitably renaming the sets) Proposition 8(1), (4) and 7(1). Specializing Theorem 1 to the case $A = B$, we arrive at Example 17.

Recall that the composite of relations $R \subseteq A \times B$ and $S \subseteq B \times C$ is a subset of $A \times C$, namely $R \otimes S = \{\langle a,c\rangle \in A \times C \mid \exists_{b\in B}\ \langle a,b\rangle \in R \text{ and } \langle b,c\rangle \in S\,\}$.

THEOREM 1: (The bi-category of sets, relations and inclusions is bi-closed w.r.t. $\otimes$)

(1) *If $S \subseteq B \times C$, then for each set A the function $\boldsymbol{P}(A \times B) \xrightarrow{-\otimes S} \boldsymbol{P}(A \times C)$ is coadjoint and its (uniquely determined) adjoint $\boldsymbol{P}(A \times C) \xrightarrow{-\triangleleft S} \boldsymbol{P}(A \times B)$ maps $T \subseteq A \times C$ to $T \triangleleft S := \{ \langle a,b \rangle \in A \times B \mid \forall_{c \in C} \ \langle a,c \rangle \in T \Leftarrow \langle b,c \rangle \in S \}$.*

(2) *If $R \subseteq A \times B$, then for each set C the function $\boldsymbol{P}(B \times C) \xrightarrow{R \otimes -} \boldsymbol{P}(A \times C)$ is coadjoint and its (uniquely determined) adjoint $\boldsymbol{P}(A \times C) \xrightarrow{R \triangleright -} \boldsymbol{P}(B \times C)$ maps $T \subseteq A \times C$ to $R \triangleright T := \{ \langle b,c \rangle \in B \times C \mid \forall_{a \in A} \ \langle a,b \rangle \in R \Rightarrow \langle a,c \rangle \in T \}$.*

(3) *If $T \subseteq A \times C$, then for each set B the functions $\boldsymbol{P}(A \times B) \xrightarrow{- \triangleright T} \boldsymbol{P}^{\mathrm{op}}(B \times C)$ and $\boldsymbol{P}(B \times C)^{\mathrm{op}} \xrightarrow{T \triangleleft -} \boldsymbol{P}(A \times B)$ constitute a Galois connection $\langle - \triangleright T, T \triangleleft - \rangle$.* □

To obtain a complete characterization via relations of all possible Galois connections between the power sets of A and B, respectively, we use the complementation-induced Galois connections κ_S and κ_B of Example 26 to define Galois connections $R_-{}^-$ and $R_\forall{}^\exists$ by

$$\begin{array}{ccc} \boldsymbol{P}(A)^{\mathrm{op}} & \xrightarrow{R_-{}^-} & \boldsymbol{P}(B) \\ \kappa_A \downarrow & & \uparrow \kappa_B \\ \boldsymbol{P}(A) & \xrightarrow[R_+{}^+]{} & \boldsymbol{P}(B)^{\mathrm{op}} \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \boldsymbol{P}(A)^{\mathrm{op}} & \xrightarrow{R_\forall{}^\exists} & \boldsymbol{P}(B)^{\mathrm{op}} \\ \kappa_A \downarrow & & \uparrow \kappa_B^{-1} \\ \boldsymbol{P}(A) & \xrightarrow[R_\exists{}^\forall]{} & \boldsymbol{P}(B) \end{array}$$

Notice that $(R_\forall{}^\exists)^{\mathrm{op}} = (R^{\mathrm{op}})_\exists{}^\forall$. Writing R^{c} for the complement $(A \times B) - R$, we also have

$$(R^{\mathrm{c}})_-{}^- = \kappa_B \circ R_\forall{}^\exists = R_\exists{}^\forall \circ \kappa_A \qquad \text{and} \qquad (R^{\mathrm{c}})_+{}^+ = \kappa_B^{-1} \circ R_\exists{}^\forall = R_\forall{}^\exists \circ \kappa_A^{-1} \tag{5}$$

as well as

$$T \triangleleft S = (T^{\mathrm{c}} \otimes S^{\mathrm{op}})^{\mathrm{c}} \qquad \text{and} \qquad R \triangleright T = (R^{\mathrm{op}} \otimes T^{\mathrm{c}})^{\mathrm{c}} \tag{6}$$

PROPOSITION 9: *A relation $R \subseteq A \times B$ satisfies*

(1) *$R^\exists(U) \subseteq R^\forall(U)$ (resp. $R^\forall(U) \subseteq R^\exists(U)$) for every $U \subseteq A$ iff each $a \in A$ is R-related to at most one (resp. at least one) $b \in B$; in this case we call R* **right unique** *(resp.* **left total***);*

(2) *$R_\exists(V) \subseteq R_\forall(V)$ (resp. $R_\forall(V) \subseteq R_\exists(V)$) for every $V \subseteq B$ iff each $b \in B$ is R-related to at most one (resp. at least one) $a \in A$; in this case we call R* **left unique** *(resp.* **right total***).* □

EXAMPLE 29: When a partial function $A \xrightarrow{h} B$ is identified with its graph, i.e., $h \subseteq A \times B$, then the axiality $h_\exists{}^\forall$ is precisely the Galois connection of Example 2, and the axiality $h^\exists{}_\forall$ is precisely the Galois connection of Example 3. From Example 2 it can be seen that $h^\exists \subseteq h^\forall$, i.e., $h^\exists(V) \subseteq h^\forall(V)$ for all $V \subseteq B$. This property in fact characterizes partial functions: by Proposition 9 $R \subseteq A \times B$ is the graph of a partial function $A \rightharpoonup B$ iff $R^\exists \subseteq R^\forall$. Moreover, R is the graph of a function from A to B iff $R^\exists = R^\forall$. □

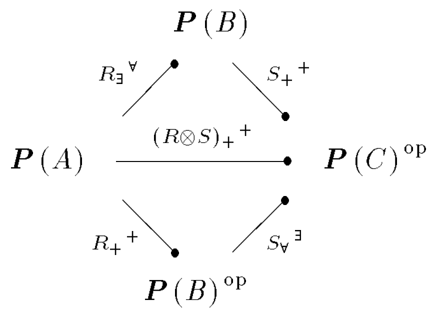
Proposition 10: *Using the backwards composition of relations $S \circ R = R \otimes S$ (as in the case of functions), one has $(S \circ R)_\exists{}^\forall = S_\exists{}^\forall \circ R_\exists{}^\forall$. Hence the axialities on $\boldsymbol{P}(A)$ form a quantale that is isomorphic to the quantale $\langle \boldsymbol{P}(A \times A), \circ \rangle$ of all relations on A (cf. Example 17).* □

Example 30: (cf. Example 15) If $\mathscr{A} = \langle A, \cdot \rangle$ is a partial semigroup and $R, S \subseteq A$, then both of $\langle R \otimes -, R \backslash - \rangle$ and $\langle - \otimes S, -/S \rangle$ are axialities from $\boldsymbol{P}(A)$ to itself. The first is induced by the relation $R' \subseteq A \times A$ given by $\langle a, b \rangle \in R'$ iff $b = r \cdot a$ for some $r \in R$, and the second is induced by the relation $S' \subseteq A \times A$ given by $\langle a, b \rangle \in S'$ iff $b = a \cdot s$ for some $s \in S$. □

Example 31: Suppose R is an **idempotent** relation on $A = B$, i.e., $R = R \otimes R$. This is tantamount to saying that R is transitive and has the **interpolation property**: for $\langle a, c \rangle \in R$ there exists some b with $\langle a, b \rangle \in R$ and $\langle b, c \rangle \in R$. Then, by Propositions 10 and 5, the axiality $R_\exists{}^\forall$ has the property that both parts are idempotent functions. Hence, restriction and corestriction to the Alexandroff topology $\mathscr{A}_R = \{ U \subseteq A \mid R_\exists(U) \subseteq U \} = \{ U \subseteq A \mid U \subseteq R^\forall(U) \}$ (cf. Example 20) yield a Galois connection $\mathscr{A}_R \xrightarrow{\pi} \mathscr{A}_R$ with the property that π_* is an interior operation and π^* is a closure operation. Since $\mathscr{A}_R$ is closed under arbitrary unions and intersections, it is a completely distributive lattice as are the images under π_* and π^*. Thus *for any idempotent relation R, the $R_\exists{}^\forall$-open (= π-open) sets and the $R_\exists{}^\forall$-closed (= π-closed) sets form isomorphic completely distributive lattices* (cf. [48] and [6]). Passing to the opposite relation, we obtain two further completely distributive lattices that are dually isomorphic to the first ones, via complementation: the $R_\exists{}^\forall$-open (closed) sets are the complements of the $R^\exists{}_\forall$-closed (open) sets.

These arguments apply, e.g., to the so-called **way-below relation** $\ll$ of a **continuous poset**. This relation is known to be idempotent, and the $\ll_\exists{}^\forall$-open sets are precisely the Scott-open sets (cf. [25], [19], and Example 12). Hence the Scott topology of a continuous poset is not only completely distributive but also dually isomorphic to the lattice of way-below sets, i.e., $\ll^\exists{}_\forall$-open sets. See also [33]. □

We conclude our primer on Galois connections with the following observation. Given two relations $R \subseteq A \times B$ and $S \subseteq B \times C$, there are at least three natural ways to produce a polarity from $\boldsymbol{P}(A)$ to $\boldsymbol{P}(C)$ by using the polarities and axialities induced by R and S, as indicated by the following diagram:



Explicitly, $S_+{}^+ \circ R_\exists{}^\forall$ comes from the relation $(R \otimes S^{\mathrm{c}})^{\mathrm{c}} = R^{\mathrm{op}} \triangleright S$, and $S_\forall{}^\exists \circ R_+{}^+$ from the relation $(R^{\mathrm{c}} \otimes S)^{\mathrm{c}} = R \triangleleft S^{\mathrm{op}}$ (use Proposition 10 and formula (6)). In general, all three polarities may be different, being induced by different relations.

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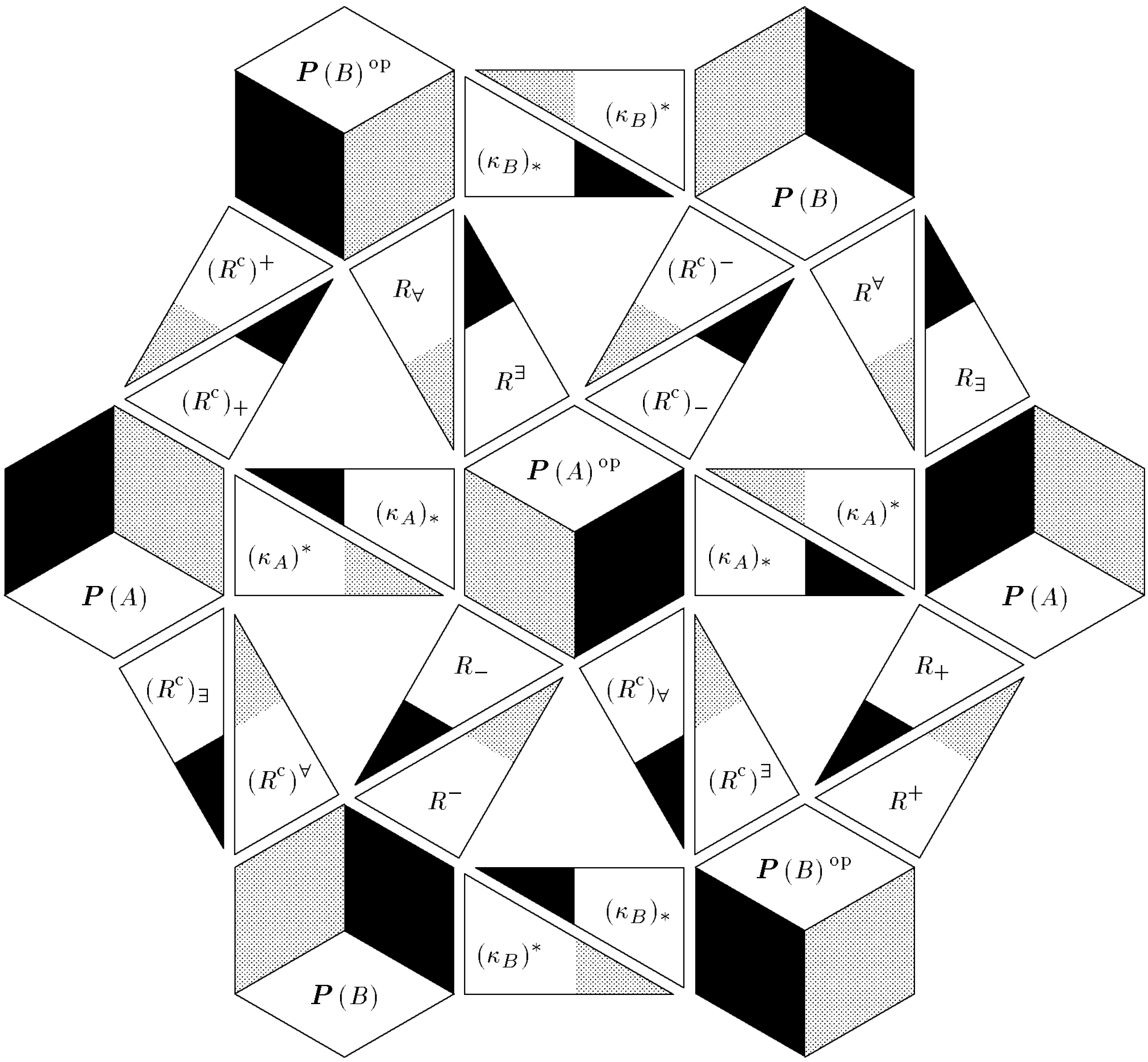
Institut für Mathematik, Universität Hannover, D-30167 Hannover, GERMANY
Department of Mathematics and Department of Computing and Information Sciences,
Kansas State University, Manhattan, KS 66506, U.S.A.
E-mail addresses: aberne@dhvrrzn1.uni-hannover.dbp.de,
koslowj@math.ksu.edu, austin@cis.ksu.edu, strecker@math.ksu.edu



A regular tessellation by Galois connections arising from one relation